Anosov components of triangle reflection groups in rank 2

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Motivation

The dynamics of **Fuchsian** and **quasi-Fuchsian** representations have been studied extensively in rank 1. We want a concrete understanding of **higher rank Anosov components**, and in particular of any boundary to these components.

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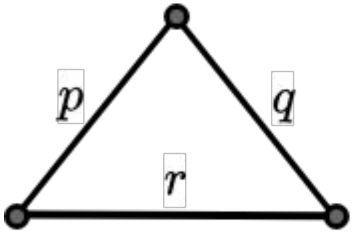
- Can representations of triangle reflection orbifolds give us insight into representations of surface groups?
- **Can computer experiments improve our understanding of the components?**

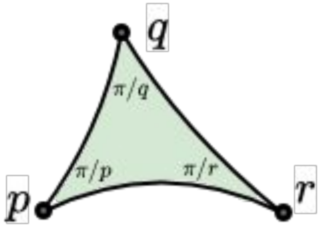
Hyperbolic triangle reflection groups

Triangle reflection groups describe the reflections of a hyperbolic triangle.

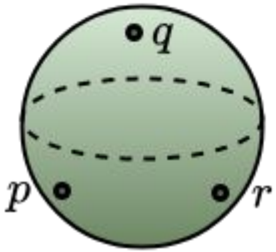
$$T(p,q,r) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (ac)^q = (bc)^r = 1 \rangle$$



Coxeter diagram



$T(p,q,r)$



$S(p,q,r)$

The subgroup which preserves orientation is called a **triangle rotation group.**

$$S(p,q,r) = \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle$$

Representations in PGL(2,C)

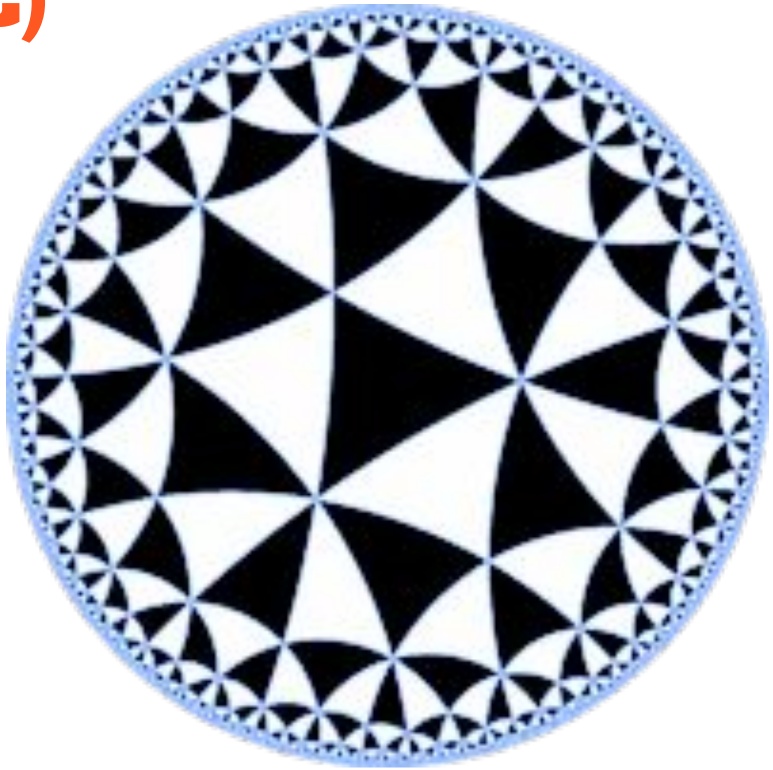
Rigidity!

There's only one Fuchsian representation of T(p,q,r) up to conjugation.

$$\rho : T(p,q,r) \to PGL(2,\mathbb{R})$$

Character varieties of S(p,q,r) have 2x the dimension of T(p,q,r) components.

Image from https://en.wikipedia.org/wiki/Triangle_group



$T(3,4,4)$

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- **Downs `23** - dimension of $G_2^{3,4}$ Hitchin components

$\left.\right\}$ **Triangle rotation groups**

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- **Porti `23** - dimensions of all 2-orbifold character varieties.

My favorite triangle reflection groups

The following triangle reflection groups have 1-dimensional Hitchin components for all three rank 2 simple adjoint Lie groups.

$T(3,4,4)$ $T(3,4,5)$ $T(3,5,5)$

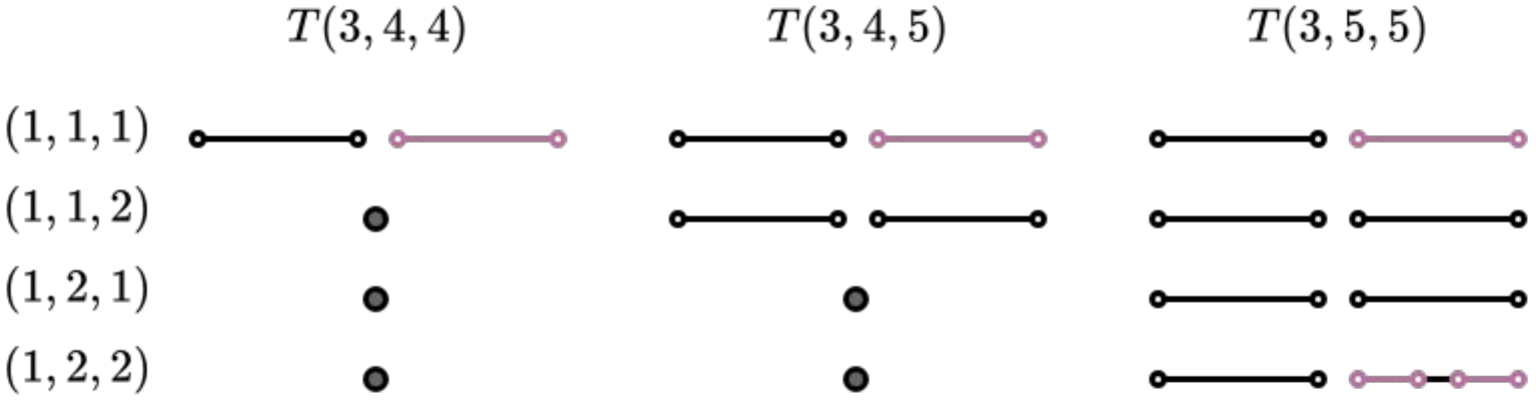
Proof: Consider $T(p,q,r)$ where $p \leq q \leq r$.
In $SL(3,\mathbb{R})$, if $p = 2$ then $\dim\,(Hitch) = 0$. (Lee, Lee, Stecker)

In $Sp(4,\mathbb{R})$, if $p > 3$ then $\dim\,(Hitch) > 1$.
If $q \leq 3$ then dim(Hitch) $= 0$. (Weir)

In $G_2^{4,3}$, if $p > 2$ and $r > 5$ then $\dim\,(Hitch) > 1$. (Downs)
So $p = 3$, and q, r are each one of 4 or 5.

Anosov components in SL(3,R)

Sometimes several Anosov components lie within a single character variety component.



(from Lee, Lee, Stecker)

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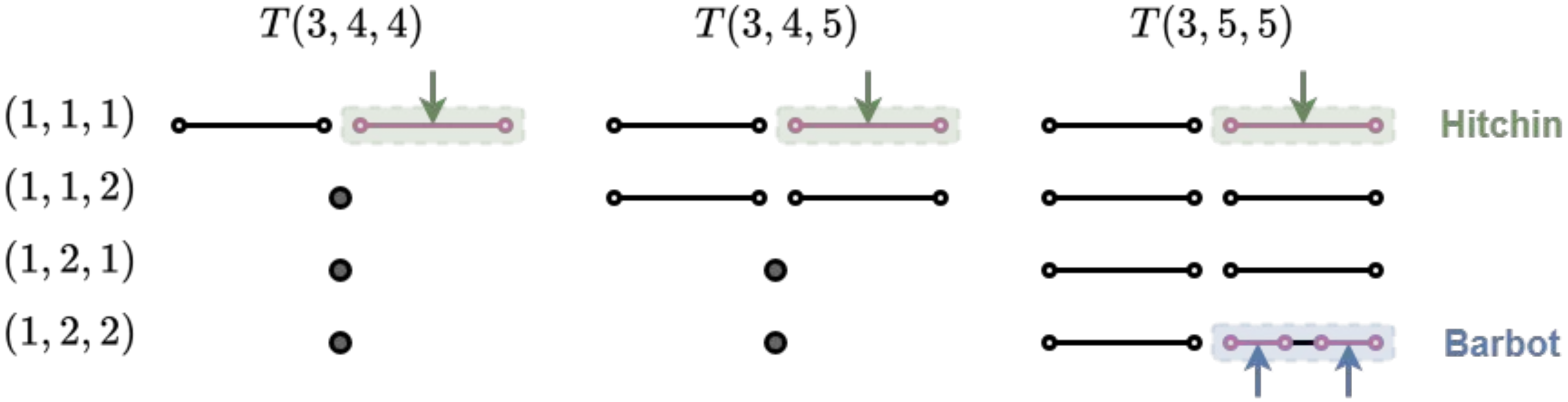
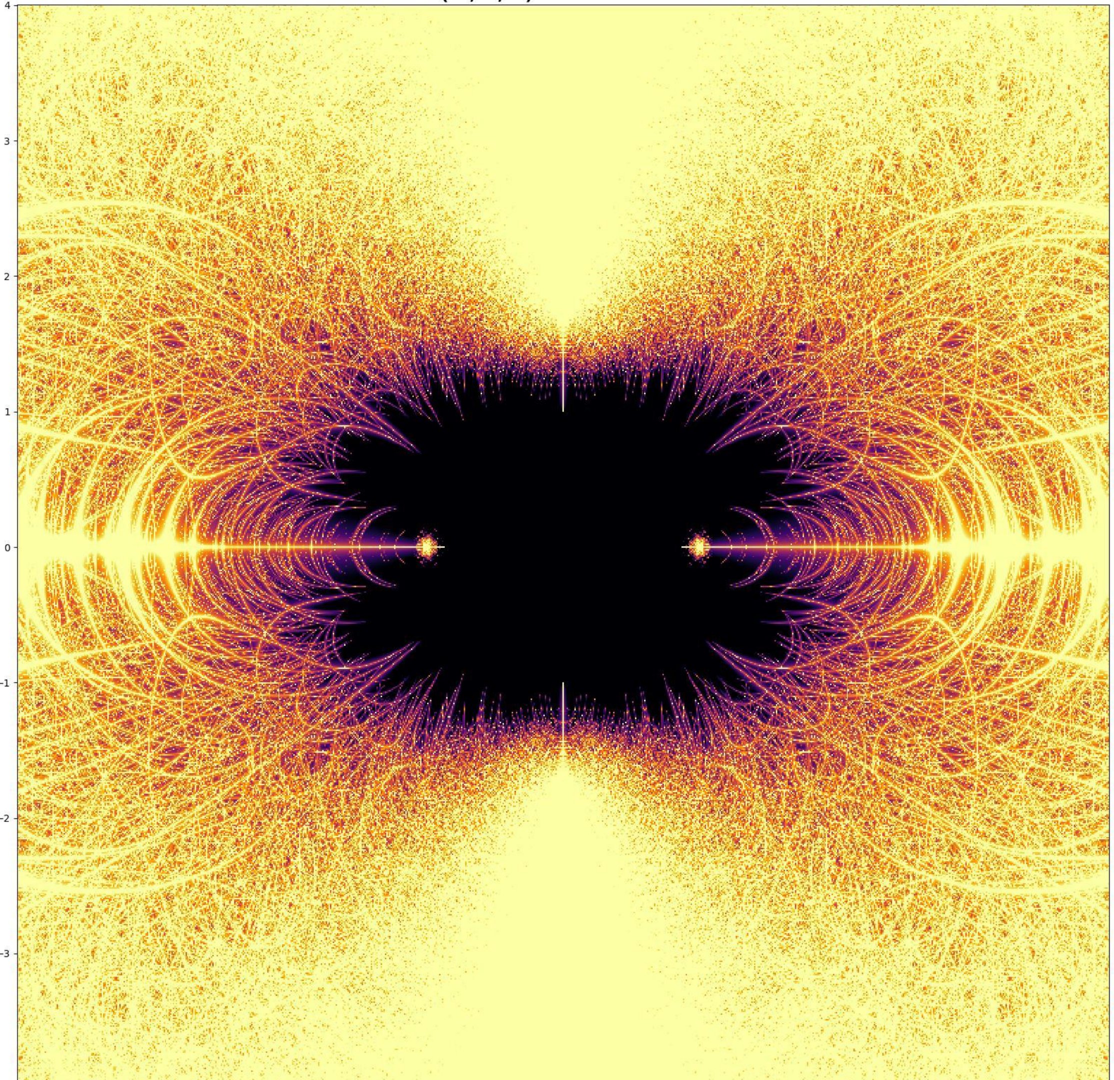


(from Lee, Lee, Stecker)

Why complexify?

- Nontrivial Anosov boundary phenomena.
- Distinct components over ***R*** may connect over **C**.
- Apply results involving holomorphic parameterization.
  e.g. **Martin-Baillon `20**

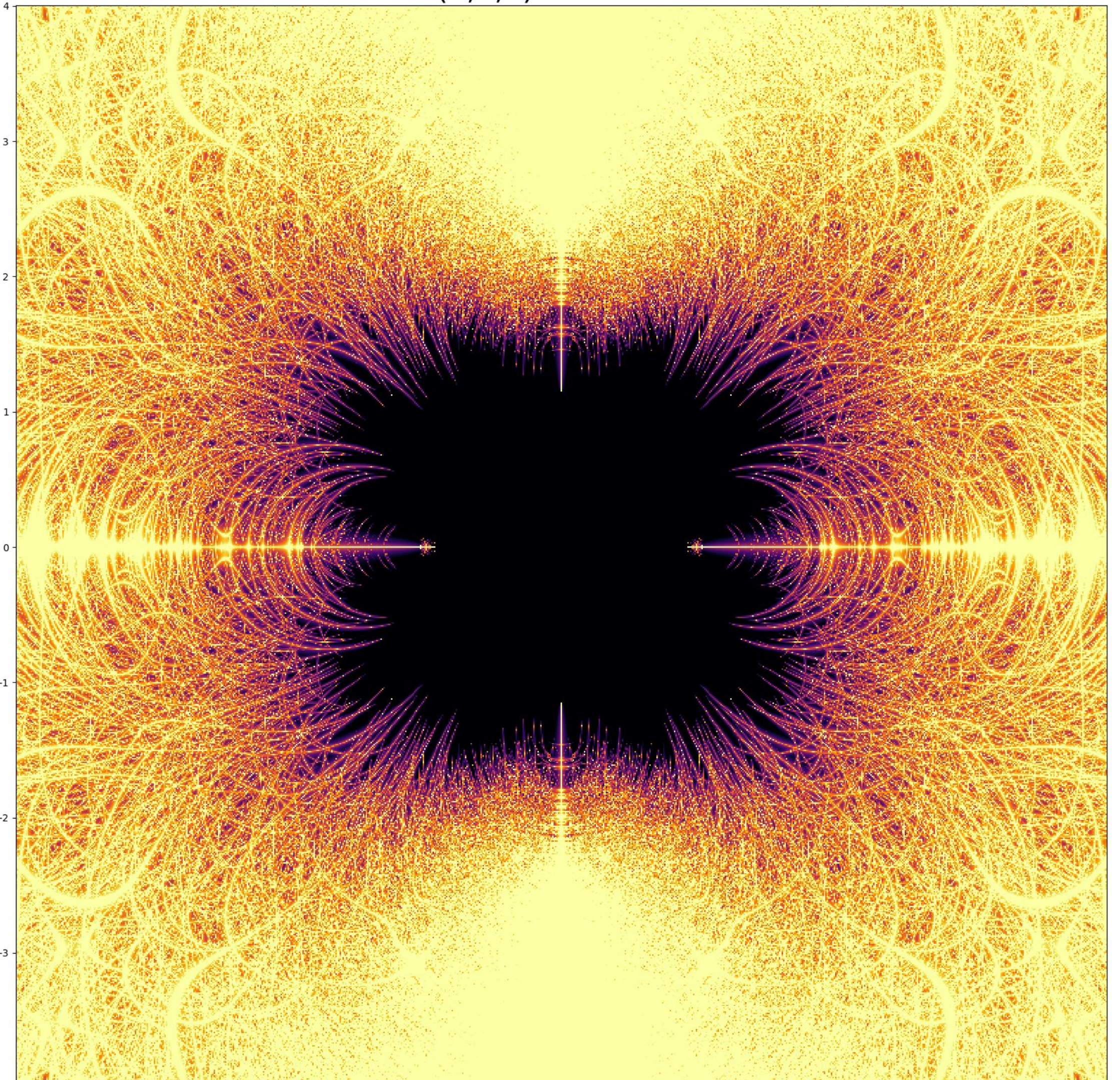


T(3,4,4) Hitchin

Black = Likely Anosov

Purple = Unlikely Anosov

Yellow/White = Not Anosov



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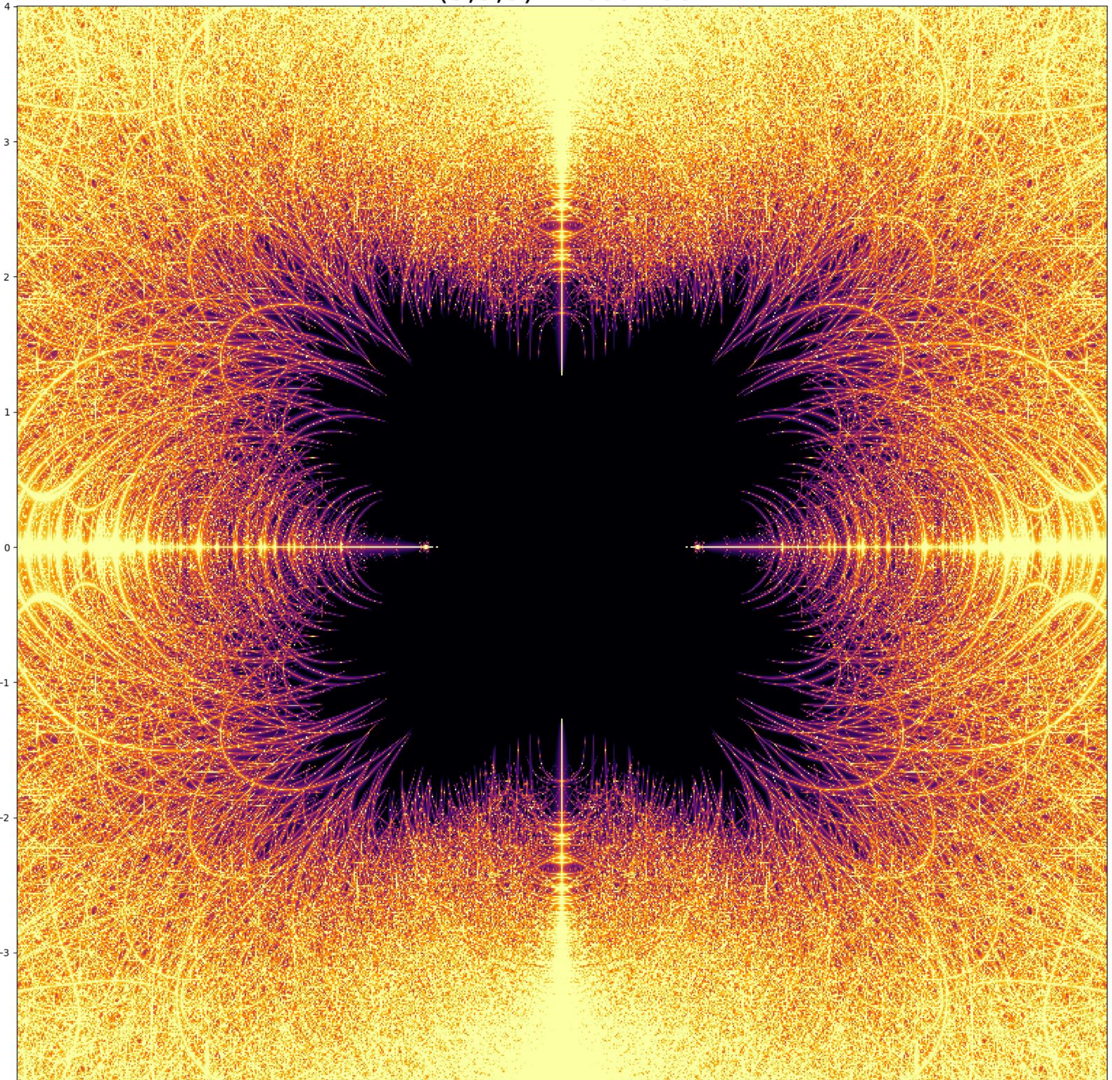

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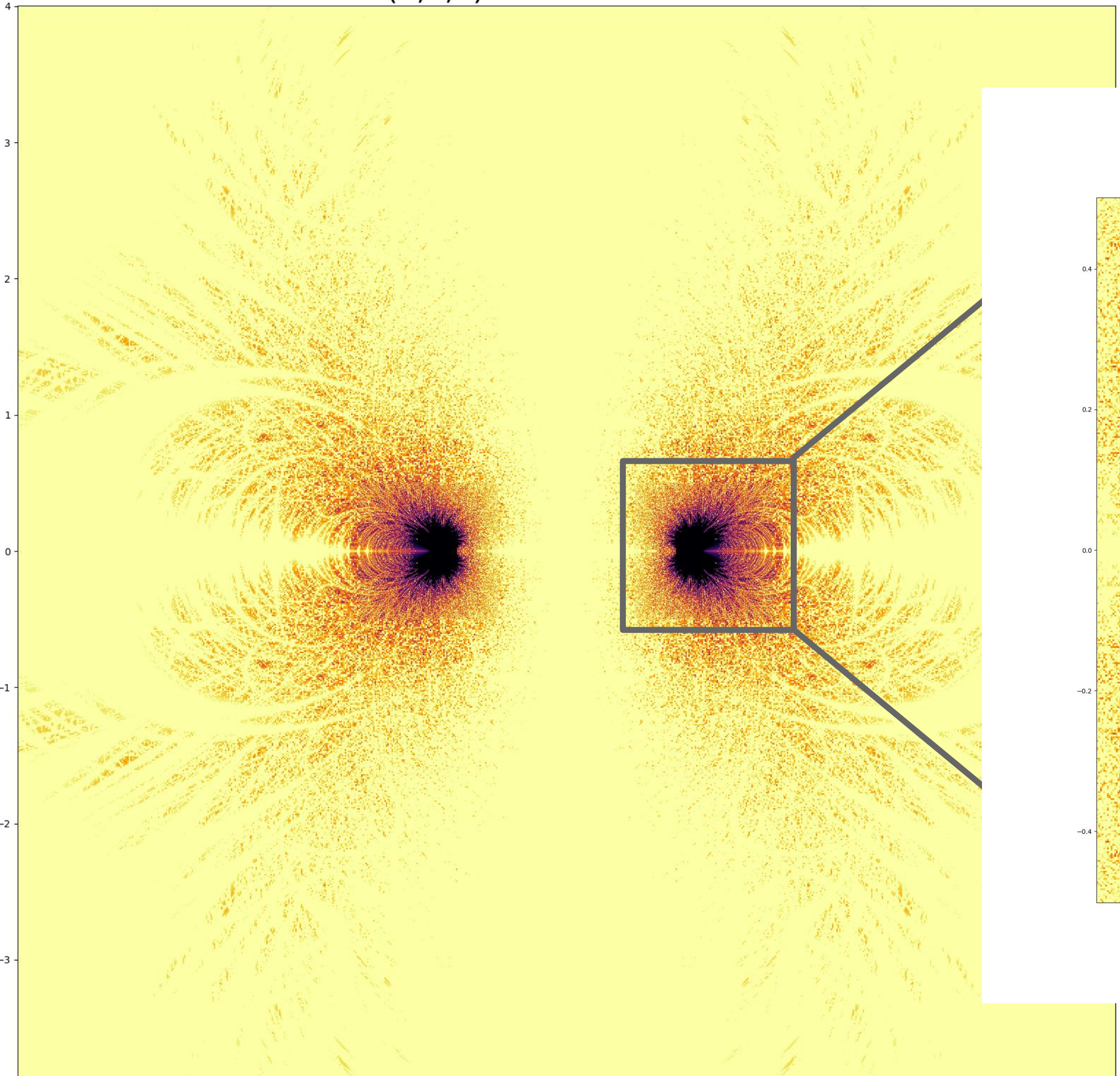
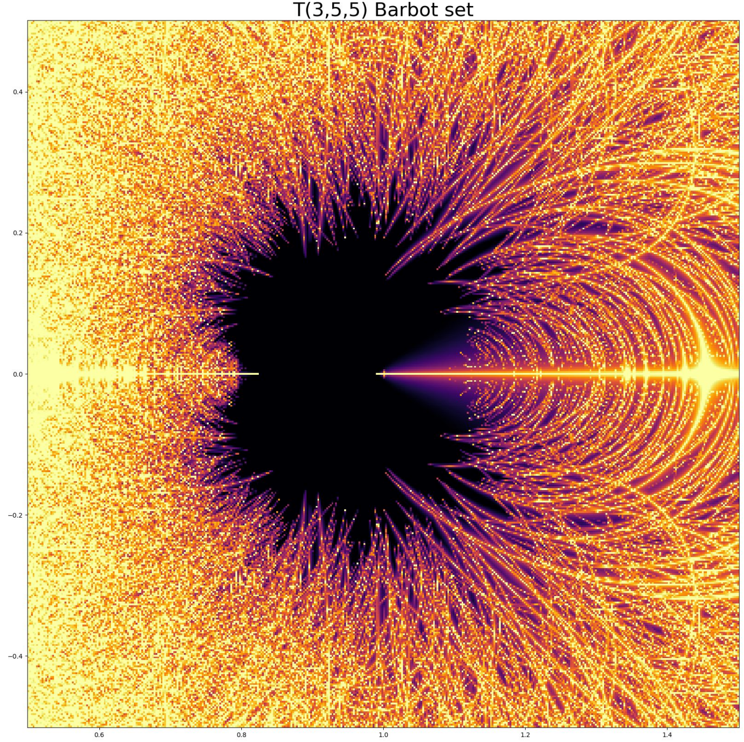
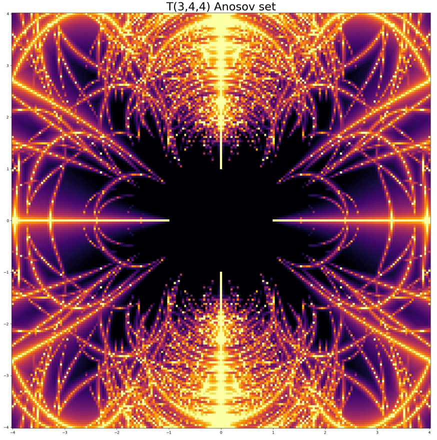


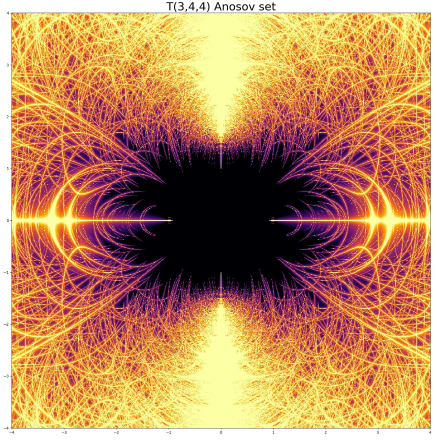


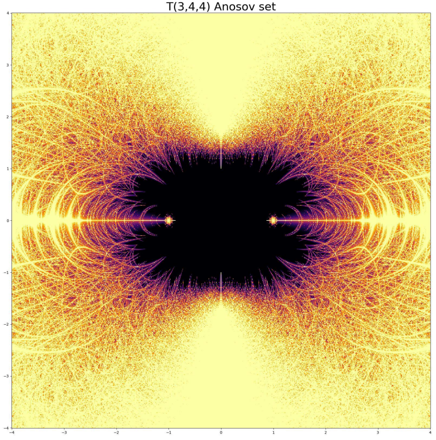
T(3,5,5) Barbot

What does "likely Anosov" mean?

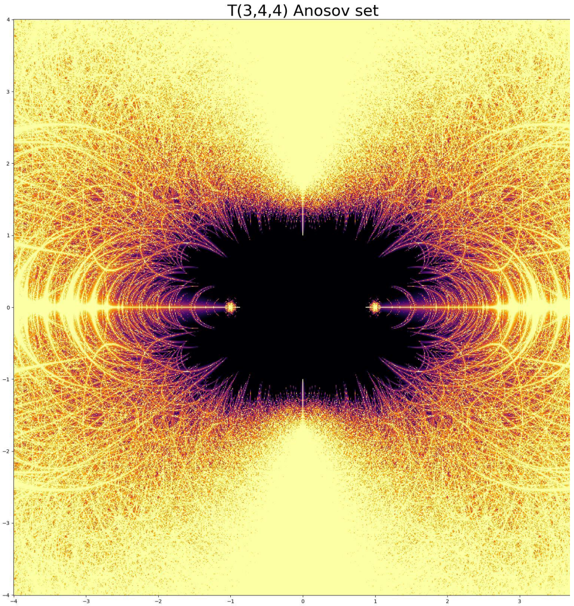
- Generate list of infinite-order elements.
- Complexify the parameterization provided by [LLS].
- Compute the minimum eigenvalue spacing.



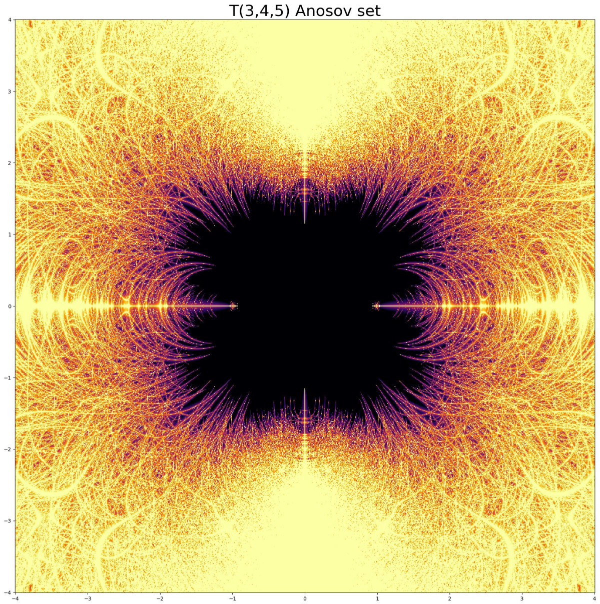




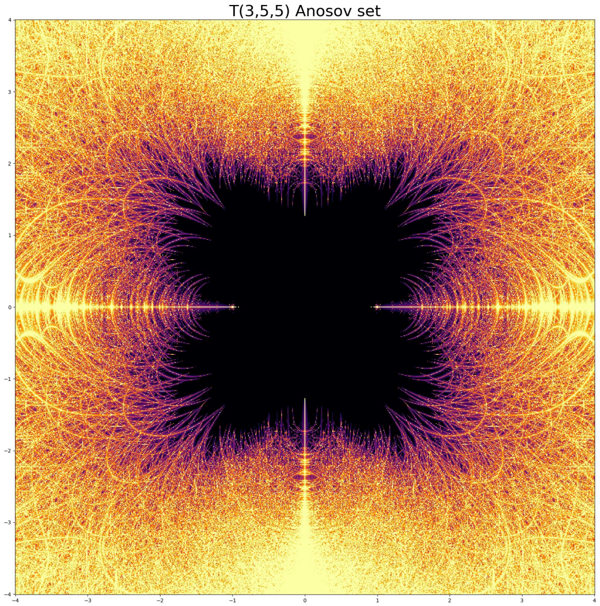
Observations in SL(3,C) and next steps



$T(3,4,4)$



$T(3,4,5)$



$T(3,5,5)$

References

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- Long, Thistlethwaite **The dimension of the Hitchin component for triangle groups** (2018)
- Elise Weir **The dimension of the restricted Hitchin component** (2020)
- Hannah Downs **The G2- Hitchin component for triangle groups** (2023)
- Lee, Lee, Stecker **Anosov triangle reflection groups in *SL(3,R)*** (2021)
- Joan Porti **Dimension of representation and character varieties for two and three-orbifolds** (2023)
- Florestan Martin-Baillon **Lyapunov exponents and stability properties of higher rank representations** (2020)
- Emily Dumas **Visualizing complex Anosov families** dumas.io/cxa/ (2023)