Hyperconvex Representations and Richardson Varieties

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DISSERTATION

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Accessibility Statement

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Dedication

In memory of my grandmother Martha, who shared her love of mathematics with me and many others.

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List of Abbreviations

- G : a complex semisimple Lie group with rank $n \geq 2$.
- $\mathfrak{g}$: a complex semisimple Lie algebra with rank $n \geq 2$.
- $\mathfrak{t}$: a Cartan subalgebra $\mathfrak{t} < \mathfrak{g}$.
- $\mathfrak{a}$: the Cartan subspace of $\mathfrak{g}$.
- $\overline{\mathfrak{a}^+}$: the closed Weyl chamber in $\mathfrak{a}$.
- N_G : Normalizer in G.
- C_G : Centralizer in G.
- $N_{\mathfrak{g}}$: Normalizer in $\mathfrak{g}$.
- $\mathcal{R}$: the roots $\mathcal{R} \subset \mathfrak{t}^*$.
- Δ : the positive simple roots as a subset of $\mathcal{R}$
- r_k : the k-th positive simple root $r_k \in \Delta$.
- B : the standard Borel subgroup of G.
- P : a standard parabolic subgroup of G.
- $\mathcal{F}$: the flag variety G/B.
- $SL_n\mathbb{C}$: the special linear group of $\mathbb{C}^n$.
- W : the Weyl group of G.
- s_k : a generating reflection of W.
- $\mathrm{Sym}(n)$: the symmetric group on n letters.
- $l(w)$: the length of a minimal word of $w \in W$.
- e : the identity element of a group.
- w_0 : the maximal length element of W.
- $\geq$: the Bruhat order on W.
- $\geq_L$: the weak left-Bruhat order on W.
- α_w : the Schubert variety $\overline{BwB}$.
- $\alpha_w(x)$: the Schubert variety $g \cdot \overline{BwB}$, where $x = gB$.
- $\alpha_k(x)$: the Schubert variety $\alpha_{s_k}(x)$.
- $w(x, y)$: the relative position of $x, y \in \mathcal{F}$.
- $\mathrm{Rich}_{x,y}$: the Richardson varieties of transverse $x, y \in \mathcal{F}$.
- ι : the Bruhat order reversing involution $\iota(w) = ww_0$.
- x^y_w : the Richardson 0-variety $\alpha_w(x) \cap \alpha_{\iota(w)}(y)$.
- x^y_k : the Richardson 0-variety $x^y_{s_k}$.
- $\mathbb{CP}^n$: the complex projective n-space.
- $\mathrm{span}(p, q)$: a Richardson 1-variety homeomorphic to $\mathbb{CP}^1$.
- $S^{\mathbb{N}}$: sequences of the form $\{x(i)\}_{i=1}^{\infty}$ where $x(i) \in S$.
- Γ : a word-hyperbolic group.
- $\partial\Gamma$: the visual boundary of Γ.
- ρ : a representation of Γ in G.
- ξ : the boundary map associated to an Anosov represention ρ.

- Λ : the image of ξ in the flag variety.
- $\mathrm{Hom}(\Gamma, G)$: the space of homomorphisms of Γ into G.
- $\mathrm{Hyp}(\Gamma, G)$: the space of hyperconvex representations of Γ into G.
- $R_x^{y,z}$: the intersection of $\mathrm{Rich}^0_{x,y}$ and $\mathrm{Rich}^0_{x,z}$.
- $I_x^{y,z}$: the Bruhat ideal $\{w \in W : x_w^y \in R_x^{y,z}\}$.
- I : a Bruhat ideal $I \subset W$.
- $I^\perp$: the perpendicular operator $w_0(W \setminus I)$.
- $Th^I(x)$: the I-thickening of a flag x.
- $Th^I(\Lambda)$ the I-thickening of the limit set Λ.
- F : a topological conjugacy.
- I_Δ : the Bruhat ideal $\{Id, s_1, \ldots, s_n\}$.
- $Gr(k, n)$: the Grassmanian of k-spaces in $\mathbb{C}^n$.
- $\wedge\mathbb{C}^n$: the space of k-forms in $\mathbb{C}^n$.
- $\mathbb{P}(V)$: the projectivization of a complex vector space V.
- $\mathrm{Sym}^k(V)$: the symmetric k-product (unordered k-tuples) of V.

Summary

In this thesis, we explore the dynamics of hyperconvex and Anosov representations. We propose and justify a definition for hyperconvex representations in complex semisimple Lie groups. We show that topological conjugacies distinguish hyperconvex representations from non-hyperconvex Anosov representations. In particular, Richardson varieties describe the dynamics of proximal elements acting on the flag variety, and topological conjugacies match the Richardson varieties associated to flags in the limit set via a Bruhat order automorphism. We identify a weaker partial topological conjugacy which also distinguishes hyperconvex representations, and construct nontrivial examples explicitly.

CHAPTER 1

Introduction

1. Classical Teichmuller theory

Let $\Gamma = \pi_1\Sigma$ be the fundamental group of a closed oriented surface Σ. Then each hyperbolic structure on Σ corresponds to an action of Γ by isometries on $\mathbb{H}^2$, which we can identify with the universal cover of Σ. Thus a hyperbolic structure can be represented by a discrete and faithful homomorphism

$$\rho_F : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^2) \cong PSL_2\mathbb{R}. \tag{1}$$

We call such a representation *Fuchsian*. The collection of Fuchsian representations of Γ (up to conjugation by $PSL_2\mathbb{R}$) is denoted $F(\Gamma)$. The set of isometry classes of marked hyperbolic structures on Σ, called the *Teichmuller space* $T(\Sigma)$ may be identified with $F(\Gamma)$.

We may embed $\mathbb{H}^2$ (and its boundary $\mathbb{RP}^1 \cong S^1$) into $\mathbb{H}^3$ (with boundary $\mathbb{CP}^1 \cong S^2$), and then consider a faithful representation

$$\rho_Q : \Gamma \to Isom^+(\mathbb{H}^3) \cong PSL_2\mathbb{C}, \tag{2}$$

which preserves a ρ_Q-invariant Jordan limit curve $\Lambda \cong S^1 \subset \mathbb{CP}^1$, and the ρ_Q-invariant cocompact domain of discontinuity

$$\Omega \sqcup \overline{\Omega} = \mathbb{CP}^1 \setminus \Lambda, \tag{3}$$

which is homeomorphic to two disjoint open discs. We call such representations *quasi-Fuchsian*, and note that they are discrete. The collection of quasi-Fuchsian representations of Γ (up to conjugation by $PSL_2\mathbb{C}$) is denoted $QF(\Gamma)$. The Bers Simultaneous Uniformization Theorem [**5**] states that

$$QF(\Gamma) \cong T(\Sigma) \times T(\overline{\Sigma}), \tag{4}$$

such that the Fuchsian locus $F(\Gamma)$ consists of elements of the form $(\rho_F, \overline{\rho_F})$.

Quasi-Fuchsian representations make up an open set of the character variety (Sullivan [**36**]). All Fuchsian and quasi-Fuchsian representations of Γ are topologically conjugate

[**36**]. An area of research sometimes referred to as higher Teichmuller theory has emerged to identify higher-rank representations which have some analogous geometric properties to Fuchsian and quasi-Fuchsian representations.

2. Anosov representations

Labourie defined Anosov representations of hyperbolic surface groups into $PSL_n\mathbb{R}$ (Labourie [**29**]). A key feature of Anosov representations $\rho : \pi_1\Sigma \to PSL_n\mathbb{R}$ is that they admit a continuous, ρ-equivariant boundary map into the full flag variety $\xi : S^1 \to \mathcal{F}(\mathbb{R}^n)$. Anosov representations are an open set of $\mathrm{Hom}(\Gamma, PSL_n\mathbb{R})$, so small perturbations of Anosov representations are also Anosov [**29**].

This property allows us to construct higher-rank Anosov representations from Fuchsian representations. Examples of Anosov representations in $SL_3\mathbb{R}$ include Hitchin and Barbot representations. Hitchin representations are those in the same connected component of $\mathrm{Hom}(\pi_1\Sigma, SL_3\mathbb{R})$ as a $PSL_3\mathbb{R}$-Fuchsian representation, e.g. of the form $\rho = i_3 \circ \rho_F$ where $i_3 : PSL_2\mathbb{R} \to SL_3\mathbb{R}$ is an irreducible embedding and ρ_F is Fuchsian (Hitchin [**26**]). Barbot described certain Anosov perturbations of $\rho = \Delta \circ \rho_F$ where $\Delta : SL_2\mathbb{R} \to SL_3\mathbb{R}$ is a reducible embedding. The Anosov component containing Barbot representations is a proper subset of a connected component of $\mathrm{Hom}(\pi_1\Sigma, SL_3\mathbb{R})$ (Barbot [**2**]).

The notion of Anosov representations has been expanded and studied in greater generality, (see e.g. [**20**]). In this more general setting one considers $\rho : \Gamma \to G$ where Γ is word-hyperbolic and G is a real or complex reductive Lie group. (Guéritaud, Guichard, Kassel, Wienhard [**22, 20, 23**]). In the case where G is a semisimple Lie group, such representations admit a continuous, ρ-equivariant, and dynamics preserving boundary map

$$\xi : \partial\Gamma \to \mathcal{F}, \tag{5}$$

where $\mathcal{F} \cong G/B$ is the full flag variety of G.

Anosov representations have emerged as the most promising higher-rank analogue of Fuchsian and quasi-Fuchsian representations [**37**].

3. Hyperconvex representations

Hyperconvex representations are a setting where additional techniques of classical Teichmuller theory apply, and thus are another compelling candidate for a higher rank analogue of quasi-Fuchsian representations.

Recent results use the following definition for hyperconvexity. Let $\rho : \Gamma \to SL_n\mathbb{C}$ be an Anosov representation with limit set $\Lambda \subset \mathcal{F}(\mathbb{C}^n)$, which is the image of the boundary map ξ. A flag $x \in \mathcal{F}(\mathbb{C}^n)$ consists of a tuple of nested linear subspaces of $\mathbb{C}^n$ of dimensions $1, \dots, n-1$ respectively, i.e.

$$x = (x_1, \dots, x_{n-1}) \tag{6}$$

where $x_i \subset x_j$ for any $i \leq j$. We adopt the convention $x_0 = \{0\}$ and $x_n = \mathbb{C}^n$.

Definition 1.1 (Hyperconvex [**18**]). Then ρ is *hyperconvex* if for all $x, y, z \in \Lambda$ and all $1 \leq k \leq n-1$ we have

$$(x_{k-1} + y_{n-k} \cap x_{k+1}) \cap (x_{k-1} + z_{n-k} \cap x_{k+1}) = x_{k-1}. \tag{7}$$

The set of hyperconvex representations $\mathrm{Hyp}(\Gamma) \subset \mathrm{Hom}(\Gamma)$ is open (Pozzeti, Sambarino, Wienhard [**32**]). Among real representations, hyperconvex and non-hyperconvex Anosov representations typically lie in different Hom components, e.g. in $SL_n\mathbb{R}$ only Hitchin representations are hyperconvex (Guichard [**21**]). Among complex representations, it is not known whether (fully) hyperconvex representations and non-hyperconvex representations lie in the same (Borel) Anosov components.

Hyperconvex representations admit continuous, injective limit set projections to certain rational curves in the flag variety isomorphic to $\mathbb{CP}^1$ [**32**]. This has allowed researchers to apply classical complex analytic results from the quasi-Fuchsian theory in higher rank. In particular, Farre, Pozzetti, and Viaggi describe Ahlfors-Bers parameters for the hyperconvex locus in $SL_n\mathbb{C}$ which, when ρ is $SL_n\mathbb{C}$-quasi-Fuchsian, aligns with the Bers parameters [**18**], and they show that hyperbolic groups Γ which admit hyperconvex representations are

virtually isomorphic to a Kleinian group [**19**]. There is also a notion of partial hyperconvexity, which applies to certain P-Anosov representations [**32**].

There has been limited acknowledgement of hyperconvex representations into semisimple Lie groups other than $SL_{n+1}\mathbb{R}$ and $SL_{n+1}\mathbb{C}$. Sambarino, using a different definition for hyperconvex, considered the direct sum of hyperconvex representations in $SL_2\mathbb{C} \times SL_2\mathbb{C}$ [**34**]. More recently, Davalo considered hyperconvex representations in $Sp_4\mathbb{R}$ as a subset of the $SL_4\mathbb{R}$ Hitchin component, where all representations are Borel Anosov in $SL_4\mathbb{R}$ [**15**].

4. Richardson varieties

Let G be a complex semisimple Lie group of rank n and $B < G$ a Borel subgroup, and consider the flag variety $\mathcal{F} = G/B$ and Weyl group W. Then W is a finite Coxeter group generated by reflections $s_1, \ldots, s_n \in W$. The Weyl group is equipped with a Bruhat order $\geq$, a length function $l : W \to \mathbb{Z}_{\geq 0}$, and a unique longest element $w_0 \in W$.

The group G acts transitively on $\mathcal{F}$ by left multiplication, and the Weyl group acts on the Cartan subspace $\mathfrak{a} < \mathfrak{g}$ by reflection. For $w \in W$, we definte the *Schubert variety* $\alpha_w = \overline{BwB}$ where

$$\overline{BwB} = \bigcup_{u \leq w} BuB \subset G/B. \tag{8}$$

For $x = gB \in \mathcal{F}$, we define the Schubert variety centered at x to be $\alpha_w(x) = g \cdot \alpha_w$. Schubert varieties are connected projective subvarieties of $\mathcal{F}$ of complex dimension $l(w)$ [**10**]. For any flags $x, y \in \mathcal{F}$ we may define the *relative position of x and y* denoted $w(x, y) \in W$ to be the smallest w such that the Schubert variety $\alpha_w(x)$ contains y.

If x and y have relative position $w(x, y) = w_0$, we call them *transverse*. The *Richardson varieties* of transverse x and y are the (nonempty) intersections $\alpha_u(x) \cap \alpha_v(y)$ where $u, v \in W$. Richardson varieties are connected projective subvarieties of $\mathcal{F}$ of complex dimension $l(u) + l(v) - l(w_0)$ (Richardson [**33**]). Define $\iota : W \to W$ to be the Bruhat order-reversing

involution $w \mapsto ww_0$. Then the Richardson varieties of dimension 0, or Richardson 0-varieties, of x and y are in bijection with W via the map $w \mapsto x_w^y$ where:

$$x_w^y = \alpha_w(x) \cap \alpha_{\iota(w)}(y). \tag{9}$$

5. Outline of key results

In this thesis, we explore the dynamics of hyperconvex and Anosov representations.

- We show that Richardson varieties describe the dynamics of proximal elements acting on the flag variety (Chapter 3).
- We propose and justify a definition for hyperconvex representations in complex semisimple Lie groups (Chapter 4).
- We show that topological conjugacies distinguish hyperconvex representations from non-hyperconvex Anosov representations (Chapter 5).

In particular, we show that topological conjugacies match the Richardson varieties associated to the limit set via a Bruhat order automorphism. We also identify a weaker partial topological conjugacy which also distinguishes hyperconvex representations, and construct nontrivial examples.

5.1. Chapter 3: Richardson varieties describe the dynamics of proximal elements on $\mathcal{F}$. The following result is likely known, but we provide an original proof.

THEOREM 1.2 (Dynamics of proximal γ on $\mathcal{F}$). *Suppose $\gamma \in G$ is a proximal element with attracting fixed point x and repelling fixed point y. Let $p \in \mathcal{F}$, and let $u = w(x,p)$ and $v = w(y,p)$. Then under the action of γ, p is attracted to y_v^x and repelled by x_u^y, i.e.*

$$\lim_{i\to\infty} \gamma^i \cdot p = y_v^x, \text{ and} \tag{10}$$

$$\lim_{i\to\infty} \gamma^{-i} \cdot p = x_u^y. \tag{11}$$

This is Theorem 3.8. We also prove that Richardson varieties vary continuously among pairs of transverse flags in Proposition 3.7.

5.2. Chapter 4: Hyperconvexity in semisimple G. We propose the following definitions for hyperconvex and partial hyperconvex representations into any complex semisimple Lie group G.

Let $\mathrm{Rich}^0_{x,y}$ denote the Richardson 0-varieties of a transverse pair of flags $x, y \in \mathcal{F}$.

Definition 1.3. An Anosov representation $\rho : \Gamma \to G$ is *hyperconvex* if for all distinct $x, y, z \in \Lambda \subset \mathcal{F}$, the intersection of Richardson 0-varieties satisfies

$$\mathrm{Rich}^0_{x,y} \cap \mathrm{Rich}^0_{x,z} = \{x\}. \tag{12}$$

We define $S_k = \{s_j : |s_k s_j| \neq 2\}$ to be a subset of the generating set of W, and $P_k < G$ the associated parabolic subgroup. Let $\mathcal{F}_k = G/P_k$.

Definition 1.4. A P_k-Anosov representation $\rho : \Gamma \to G$ is $\{s_k\}$-*hyperconvex* if for all distinct $x, y, z \in \Lambda \subset \mathcal{F}_k$, the Richardson 0-varieties satisfy

$$x^y_{s_k} \neq x^z_{s_k}. \tag{13}$$

When G is type A_n, our definitions for hyperconvexity and partial hyperconvexity agree with Def. 1.1 and Def. 4.3, respectively (Proposition 4.5).

Such representations make up an open set of $\mathrm{Hom}(\Gamma, G)$ (Proposition 4.8), and they admit continuous, injective limit set projections of the form $\pi^x_k : \Lambda \to \alpha_k(x)$ given by $y \mapsto x^y_{s_k}$ and $x \mapsto x$ (Proposition 4.9).

We also give a detailed incidence geometry description of Richardson 0-varieties in types A, B, C, D, and G_2 (Chapter 4, Section 7).

5.3. Chapter 5: Hyperconvexity is preserved under topological conjugacy. This chapter is the primary original research contribution of this thesis.

Let $\rho, \rho' : \Gamma \to G$ be Anosov representations, with limit sets $\Lambda, \Lambda' \subset \mathcal{F}$ which are the images of the boundary maps $\xi, \xi' : \partial\Gamma \to \mathcal{F}$. Then a *topological conjugacy* between ρ and

ρ' is an equivariant homeomorphism $F : \mathcal{F} \to \mathcal{F}$, i.e. for all $x \in \mathcal{F}$ and $\gamma \in \Gamma$,

$$F(\rho(\gamma) \cdot x) = \rho'(\gamma) \cdot F(x). \tag{14}$$

Using Richardson varieties, we identify restrictions on topological conjugacies between Anosov representations. In particular, topological conjugacies preserve hyperconvexity.

THEOREM 1.5. *If ρ and ρ' are topologically conjugate, then ρ is hyperconvex if and only if ρ' is hyperconvex.*

This is Theorem 5.1. We actually prove a stronger statement in Lemma 5.2, which relies on Theorem 3.8 and Proposition 3.7:

LEMMA 1.6. *If $F : \mathcal{F} \to \mathcal{F}$ is a topological conjugacy between ρ and ρ', then there exists a Bruhat order automorphism $\phi \in Aut(W)$ such that F ϕ-matches the Richardson varieties of Λ, i.e. for any distinct $x, y \in \Lambda$ and all $u, v \in W$,*

$$F(\alpha_u(x) \cap \alpha_v(y)) = \alpha_{\phi(u)}(F(x)) \cap \alpha_{\phi(v)}(F(y)). \tag{15}$$

We also introduce a weaker notion of topological conjugacy which also preserves hyperconvexity. Let $I \subset W$ be a Bruhat order ideal. We define $Th^I(x) = \bigcup_{w \in I} \alpha_w(x)$. The limit set thickening is $Th^I(\Lambda) = \bigcup_{x \in \Lambda} Th^I(x) \subset \mathcal{F}$.

Let $\rho, \rho' : \Gamma \to G$ Anosov. Then a homeomorphism $F : Th^I(\Lambda) \to Th^I(\Lambda')$ is an *I-topological conjugacy* between ρ and ρ' if, for all $p \in Th^I(\Lambda)$ and $\gamma \in \Gamma$,

$$F(\rho(\gamma) \cdot p) = \rho'(\gamma) \cdot F(p). \tag{16}$$

All Anosov representations are $\{e\}$-topologically conjugate where $F = \xi' \circ \xi^{-1}$, and full topological conjugacies on $\mathcal{F}$ are W-topological conjugacies. Define $I_\Delta = \{e, s_1, \ldots, s_n\}$. Then I_Δ-topological conjugacies preserve hyperconvexity.

THEOREM 1.7. *If ρ and ρ' are I_Δ-topologically conjugate, then ρ is hyperconvex if and only if ρ' is hyperconvex.*

This is Theorem 5.8. We also construct explicit, nontrivial examples of I_Δ-topological conjugacies (Propositions 5.11 and 5.10).

CHAPTER 2

Background

1. Hyperbolic groups and their boundaries

This section is inspired by the exposition in Bridson and Haefliger's textbook [**9**] and Kapovich and Benakli's survey paper [**27**] and the references therein.

1.1. Hyperbolic groups. Let (X, d) be a metric space. Let $\delta \geq 0$. A geodesic triangle $\triangle(x, y, z)$ is called *δ-slim* if each side $\overline{xy}$ is contained in the δ-neighborhood of the other sides, i.e. $\overline{xy} \subset B_\delta(\overline{xz}) \cup B_\delta(\overline{yz})$. A geodesic metric space (X, d) is called *δ-hyperbolic* if all triangles are δ-slim. We call (X, d) *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Let Γ be a group with a finite generating set S. We call Γ *word hyperbolic* if the Cayley graph $\mathcal{C}(\Gamma, S)$ equipped with the word metric is hyperbolic. This property does not depend on the generating set S.

Proposition 2.1 (Proposition 2.2 pg. 448 [**9**]). *Every hyperbolic group is finitely presented.*

Example 2.2. *Examples of hyperbolic groups include:*

(1) *Any finite group, since the Cayley graph has finite diameter,*
(2) *The infinite cyclic group $\mathbb{Z} = \langle a \rangle$,*
(3) *The free group on k generators $\mathbb{F}^k = \langle a_1, \ldots, a_k \rangle$, and*
(4) *The fundamental group of a surface Σ_g of genus $g \geq 2$:*

$$\pi_1 \Sigma_g = \langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle. \tag{17}$$

Proposition 2.3 (Proposition 2.22 pg. 458 [**9**]). *If a hyperbolic group Γ is infinite, then it contains an infinite-order word.*

Throughout the remainder of this section, we assume that Γ is a word hyperbolic group.

1.2. The visual boundary $\partial\Gamma$. Next, we define the visual boundary of hyperbolic groups. We call $q : \mathbb{R}_{\geq 0} \to X$ a *quasigeodesic ray* if there exist constants $K \geq 1$ and $C \geq 0$

such that for all $s, t \in \mathbb{R}_{\geq 0}$ we have

$$\frac{1}{K}|s-t| - C \quad \leq \quad d(q(s), q(t)) \quad \leq \quad K|s-t| + C. \tag{18}$$

We call two quasigeodesic rays q, q' *asymptotic* if $\sup_{t \in \mathbb{R}_{\geq 0}} d(q(t), q'(t))$ is finite. Clearly this is an equivalence relation on quasigeodesic rays in X.

DEFINITION 2.4. The Gromov boundary $\partial\Gamma$ of Γ is the space of quasigeodesic rays in Γ up to equivalence.

$$\partial\Gamma \cong \{q : \mathbb{R}_{\geq 0} \to \Gamma\}/q \sim q' \text{ if asymptotic.} \tag{19}$$

The space $\Gamma \sqcup \partial\Gamma$ comes equipped with a natural topology such that $\Gamma \sqcup \partial\Gamma$ is a compact Hausdorff space and $\partial\Gamma$ is closed. (Proposition 3.7 pg. 429 [**9**]).

THEOREM 2.5 (Theorem 2.28 [**27**]). *Exactly one of the following occurs:*

(1) Γ *is finite and* $\partial\Gamma$ *is empty.*
(2) Γ *contains an infinite cyclic group as a subgroup of finite index and* $\partial\Gamma$ *consists of two points.*
(3) Γ *contains a subgroup isomorphic to* $\mathbb{F}^2$ *and the boundary* $\partial\Gamma$ *is an infinite perfect (i.e. without isolated points) compact metrizable space.*

Groups of the third type are called *non-elementary*, and groups of the first two types are called *elementary*.

1.3. Action of Γ on $\partial\Gamma$. The group Γ acts on itself and on its boundary.

PROPOSITION 2.6 (Proposition 4.2 [**27**]). *The action of the hyperbolic group* Γ *on its boundary* $\partial\Gamma$ *has the following properties:*

(1) *Any element $\gamma \in \Gamma$ of infinite order in Γ acts as a loxodromic isometry of the Cayley graph of Γ. There are exactly two points $\gamma^+, \gamma^- \in \partial\Gamma$ fixed by γ:*

$$\gamma^+ = \lim_{n\to\infty} \gamma^n \text{ and} \tag{20}$$

$$\gamma^- = \lim_{n\to\infty} \gamma^{-n}. \tag{21}$$

(2) *The action of Γ on $\partial\Gamma$ is minimal, i.e. for any $p \in \partial\Gamma$ the orbit $\Gamma \cdot p$ is dense in $\partial\Gamma$.*

(3) *The set of rational points $\{\gamma^+ | \gamma \in \Gamma$ is of infinite order$\}$ is dense in $\partial\Gamma$.*

(4) *The set of pole-pairs $\{(\gamma^-, \gamma^+) | \gamma \in \Gamma$ is of infinite order$\}$ is dense in $\partial\Gamma \times \partial\Gamma$.*

Suppose that for any $x, z \in \partial\Gamma$, there exists a quasigeodesic sequence $\{\gamma(i)\}_{i=1}^{\infty} \in \Gamma^{\mathbb{N}}$ such that $\lim_{i\to\infty} \gamma(i) \cdot x = x$, but $\lim_{i\to\infty} \gamma(i) \cdot y = z$ for all $y \in \partial\Gamma \setminus \{x\}$. Then we call the action of Γ on $\partial\Gamma$ a *convergence group action*. This is equivalent to saying that the action of Γ on distinct triples of $\partial\Gamma$ is properly discontinuous.

Furthermore, if the action of Γ on distinct triples of $\partial\Gamma$ is cocompsct, then we call its action a *uniform convergence group action*.

PROPOSITION 2.7 (Convergence group action [**27**]). *The action of Γ on $\partial\Gamma$ is a uniform convergence group action.*

THEOREM 2.8 (Theorem 4.3 [**27**]). *Let $\gamma \in \Gamma$ be an element of infinite order. Then for any open sets $U, V \subset \partial\Gamma$ with $\gamma^+ \in U$,$\gamma^- \in V$ there is a finite $n > 1$ such that*

$$\gamma^n(\partial\Gamma \setminus V) \subset U. \tag{22}$$

2. Complex semisimple Lie groups and algebras

This and the following sections rely on Baston and Eastwood's textbook [**4**], Hall's textbook [**24**], and content from the background sections of Dumas and Sanders [**16**] and Gueritaud, Guichard, Kassel, and Wienhard [**20**].

2.1. Semisimple Lie algebras. A *complex Lie group* G is a complex manifold with a group structure such that the map $G \times G \to G$ given by $(g, h) \mapsto gh^{-1}$ is holomorphic. Denote by $\mathfrak{g} = Lie(G)$ the *Lie algebra* of G, which is a Lie algebra over $\mathbb{C}$.

We call a subspace $\mathfrak{i} < \mathfrak{g}$ an ideal if $[\mathfrak{i}, \mathfrak{g}] < \mathfrak{i}$. We call $\mathfrak{g}$ *simple* if there are no nontrivial, proper ideals. We call $\mathfrak{g}$ *semisimple* if it is the direct sum of simple Lie algebras. We call G *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple, and we call G *simple* if it is connected and its Lie algebra is simple. Our standing assumption is that G (and hence $\mathfrak{g}$) is semisimple.

2.2. Cartan subalgebra. A maximal abelian self-normalizing subalgebra $\mathfrak{h} < \mathfrak{g}$ is called a *Cartan subalgebra*. All Cartan subalgebras of $\mathfrak{g}$ are conjugate. The *rank* of G is the (complex) dimension of any Cartan subalgebra. We assume that G has rank $n \geq 2$.

Fix some Cartan subalgebra $\mathfrak{h}$ and consider the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ given by

$$(\mathrm{ad}h)v = [h, v] \quad \forall h \in \mathfrak{h}, v \in \mathfrak{g}. \tag{23}$$

Then $\mathfrak{g}$ decomposes into a direct sum of eigenspaces for the action of $\mathfrak{h}$. The zero eigenspace is $\mathfrak{h}$. For $r \in \mathfrak{h}^*$, define

$$\mathfrak{g}_r = \{v \in \mathfrak{g} \quad : \quad [h, v] = r(h)v \quad \forall h \in \mathfrak{h}\}. \tag{24}$$

Then call $\mathcal{R}$ to be the collection of nonzero $r \in \mathfrak{h}^*$ with nontrivial eigenspace. We call $r \in \mathcal{R}$ a *root* of $\mathfrak{h}$, and $\mathfrak{g}_r$ the corresponding *root space*.

THEOREM 2.9. *The Lie algebra $\mathfrak{g}$ decomposes into its Cartan subalgebra and root spaces, i.e.*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{r \in \mathcal{R}} \mathfrak{g}_r. \tag{25}$$

We may partition $\mathcal{R}$ into *positive roots* $\mathcal{R}^+$ and *negative roots* such that $\mathcal{R}^- = -\mathcal{R}^+$ and so that $\mathcal{R}^+$ and $\mathcal{R}^-$ are separated by a hyperplane in the $\mathbb{R}$-span of $\mathcal{R}$.

This defines a *standard Borel subalgebra*

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{r \in \mathcal{R}^+} \mathfrak{g}_r. \tag{26}$$

We note that $\mathfrak{b}$ is a maximum solvable subalgebra. The *simple roots* $\Delta \subset \mathcal{R}^+$ are those that cannot be written as a sum of other positive roots. Then for any $r \in \Delta$ define $\mathcal{R}_r^- \subset \mathcal{R}^-$ the elements which can be written as negative integral sums of $\Delta \setminus \{r\}$. This defines a *maximal parabolic subalgebra*

$$\mathfrak{p}_r = \mathfrak{b} \oplus \bigoplus_{s \in \mathcal{R}_r^-} \mathfrak{g}_s. \tag{27}$$

If $S \subset \Delta$ then we define the *parabolic subalgebra*

$$\mathfrak{p}_S = \bigcap_{r \in S} \mathfrak{p}_r. \tag{28}$$

Clearly $\mathfrak{p}_\Delta = \mathfrak{b}$ and $\mathfrak{p}_\emptyset = \mathfrak{g}$. If S is such that $S \subset \Delta$ is a proper subset, then

$$\mathfrak{g} > \mathfrak{p}_S > \mathfrak{b} > \mathfrak{h}. \tag{29}$$

Define the standard parabolic subgroup of type S to be $P_S = N_G(\mathfrak{p}_S)$, the standard Borel subgroup $B = N_G(\mathfrak{b})$, and the standard Cartan subgroup $H = C_G(\mathfrak{h})$, where N_G and C_G are the normalizer and centralizer in G, respectively. As the notation suggests, the Lie algebra of P_S is $\mathfrak{p}_S$, etc. The group H is a *maximal torus* which is a maximal abelian subgroup of G, and if G is connected is isomorphic to $(\mathbb{C}^*)^n$. Then we have the containment

$$G > P_S > B > H. \tag{30}$$

We call the conjugates of P_S *parabolic subgroups*, the conjugates of B *Borel subgroups*, and the conjugates of H *Cartan subgroups* or *maximal tori*.

There is a maximal compact subgroup $K < G$ whose intersection with H is a *maximal compact torus* $K \cap T$ which is an abelian subgroup isomorphic to $(S^1)^n$. Let $\mathfrak{k}$ be its Lie algebra. Then considering the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, a *Cartan subspace* is $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{k}$,

which is a maximum real abelian subalgebra of $\mathfrak{k}$ consisting of semisimple elements. Any root $r \in \mathcal{R}$ restricted to $\mathfrak{a}$ is real-valued, and uniquely identified by this restriction, so we abuse notation and identify a root $r \in \mathfrak{h}^*$ with the corresponding restricted root $r \in \mathfrak{a}^*$ depending on context. We call $\mathfrak{a}^+$ the *positive Weyl chamber*, where

$$\mathfrak{a}^+ = \{v \in \mathfrak{a} : r(v) > 0 \quad \forall r \in \Delta\}, \tag{31}$$

and its closure $\overline{\mathfrak{a}^+}$ the *closed Weyl chamber*, where

$$\overline{\mathfrak{a}^+} = \{v \in \mathfrak{a} : r(v) \geq 0 \quad \forall r \in \Delta\}. \tag{32}$$

Let $A = \exp\overline{\mathfrak{a}^+}$, and on a group level this gives us the Cartan decomposition $G = KAK$. If $g = k\exp(v_g)k'$ for some $k, k' \in K$, then $v_g \in \overline{\mathfrak{a}^+}$ is uniquely determined. This defines the *Cartan projection*:

$$\mu : G \to \overline{\mathfrak{a}^+} \tag{33}$$

$$\mu(g) \mapsto v_g. \tag{34}$$

This map is continuous and proper [**20**]. There is also the *Jordan projection*, given by

$$\lambda : G \to \overline{\mathfrak{a}^+} \tag{35}$$

$$\lambda(g) = \lim_{n\to\infty} \frac{1}{n}\mu(g^n). \tag{36}$$

It may also be characterized by the Jordan decomposition where $g = g_h g_e g_u$ and the conjugacy class of the hyperbolic factor g_h intersects $\exp\overline{\mathfrak{a}^+}$ in a unique point $\exp(\lambda(g))$.

2.3. Classification of semisimple Lie algebras. In his celebrated 1894 thesis, Elie Cartan classified all of the simple Lie algebras using the geometry of their root systems [**14**]. There are four infinite families of complex simple Lie algebras, called the *classical Lie algebras*:

- Type A_n : $\mathfrak{a}_n \cong \mathfrak{sl}_{n+1}\mathbb{C}$ where $n \geq 1$,
- Type B_n : $\mathfrak{b}_n \cong \mathfrak{so}_{2n+1}\mathbb{C} < \mathfrak{sl}_{2n+1}\mathbb{C}$ where $n \geq 2$,

- Type C_n : $\mathfrak{c}_n \cong \mathfrak{sp}_{2n}\mathbb{C} < \mathfrak{sl}_{2n+1}\mathbb{C}$ where $n \geq 3$, and
- Type D_n : $\mathfrak{d}_n \cong \mathfrak{so}_{2n}\mathbb{C} < \mathfrak{sl}_{2n+1}\mathbb{C}$ where $n \geq 4$.

There are also five *exceptional Lie algebras*:

- Type E_6 : $\mathfrak{e}_6 < \mathfrak{sl}_{27}\mathbb{C}$,
- Type E_7 : $\mathfrak{e}_7 < \mathfrak{sl}_{56}\mathbb{C}$,
- Type E_8 : $\mathfrak{e}_8 < \mathfrak{sl}_{248}\mathbb{C}$,
- Type F_4 : $\mathfrak{f}_4 < \mathfrak{sl}_{26}\mathbb{C}$, and
- Type G_2 : $\mathfrak{g}_2 < \mathfrak{so}_7\mathbb{C} < \mathfrak{sl}_7\mathbb{C}$.

In each Lie algebra, n refers to the rank, i.e. the dimension of the Cartan subalgebra. For each simple Lie algebra, there is a smallest type-A Lie algebra in which it can be embedded, called the fundamental representation, so we can consider each simple algebra to be a subalgebra of some minimal $\mathfrak{sl}_N\mathbb{C}$. These are listed above.

If a Lie algebra $\mathfrak{g}$ is semisimple, then it is isomorphic to the direct sum of simple factors $\mathfrak{g}_1, \ldots, \mathfrak{g}_N$ such that

$$\mathfrak{g} \cong \bigoplus_{i=1}^{N} \mathfrak{g}_i. \tag{37}$$

This decomposition is unique up to permuting the order of the simple components [**24**].

If G and G' have Lie algebra $\mathfrak{g}$, then they are locally isomorphic, but not necessarily isomorphic. There are typically several semisimple Lie groups that have the Lie algebra $\mathfrak{g}$. For example, both $SL_{n+1}\mathbb{C}$ and $PSL_{n+1}\mathbb{C}$ are type A_n and have Lie algebra (isomorphic to) $\mathfrak{sl}_{n+1}\mathbb{C}$. Similarly, both $SO_{2n}\mathbb{C}$ and $O_{2n}\mathbb{C}$ are type D_n and have Lie algebra (isomorphic to) $so_{2n}\mathbb{C}$.

However, in the first case, the center $Z < SL_{n+1}\mathbb{C}$ is finite, and $PSL_{n+1}\mathbb{C} \cong SL_{n+1}\mathbb{C}/Z$. More generally, if G has a nontrivial center $Z < G$, then there is a short exact sequence

$$1 \longrightarrow Z \longrightarrow G \longrightarrow G/Z \longrightarrow 1. \tag{38}$$

In the second case, $SO_{2n}\mathbb{C}$ embeds in $O_{2n}\mathbb{C}$ as one of two connected components. The connected component containing the identity $G_0 < G$ is a normal subgroup. If G has N connected components, and G/G_0 is a finite group. Thus there is an exact sequence

$$1 \longrightarrow G_0 \to G \longrightarrow G/G_0 \longrightarrow 1. \tag{39}$$

Finally, even some classical simple Lie groups have a nontrivial fundamental group $\pi_1 G$. Then $\pi_1 G$ acts on the universal cover Lie group $\widetilde{G}$ discretely, and we have an exact sequence

$$1 \longrightarrow \pi_1 G \to \widetilde{G} \longrightarrow G \to 1. \tag{40}$$

2.4. The Weyl group. This section is inspired by the text of Bjorner and Brenti [**7**].

The *Weyl group* of G is the group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ which acts on the Cartan subspace $\mathfrak{a}$ via reflections through hyperplanes in the kernels of the simple roots. The Weyl group is thus a finite Coxeter group. Dually, w acts on $\mathfrak{a}^*$, also as a reflection group. The action of W on $\mathfrak{a}^*$ permutes the restricted simple roots, which we interpret as a group of permutations on the corresponding simple roots $\mathcal{R}$. By construction, the inclusion of $\mathfrak{a} \to \mathfrak{h}$ means there is an inclusion $N_K(\mathfrak{a}) \to N_G(H)$, which induces an isomorphism $W \cong N_G(H)/H$. It is typically clear from context whether we consider W to act on the roots or the restricted roots.

The Dynkin diagrams of each simple type are given in Figure 1. The Weyl group W is a finite Coxeter group with a presentation described by the Dynkin diagram in the following way:

- For each vertex v_i, there is a reflection generator s_i such that $s_i^2 = e$,
- if v_i and v_j are not connected, then $(s_i s_j)^2 = e$,
- if v_i and v_j are single connected, then $(s_i s_j)^3 = e$,
- if v_i and v_j are doubly connected, then $(s_i s_j)^4 = e$, and
- if v_i and v_j are triply connected, then $(s_i s_j)^6 = e$.

The Dynkin diagram of a semisimple Lie algebra consists of disjoint copies of the Dynkin diagrams of the semisimple components [**24**]. In particular, the Weyl group is isomorphic to

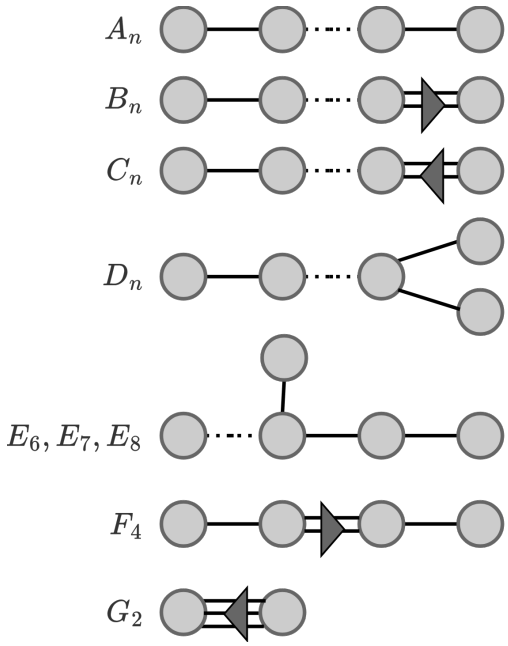

FIGURE 1. The Dynkin diagrams of the simple Lie algebras.

the direct sum of the simple component Weyl groups:

$$W \cong \bigoplus_{i=1}^{N} W_i. \tag{41}$$

Weyl groups are are equipped with a *Chevalley-Bruhat order*. We identify the set of generating reflections in W with the set of simple roots in $\mathcal{R}$, and refer to both as Δ. The length function $l : W \to \mathbb{Z}_{\geq 0}$ is given by the length of a minimal word for w under the generating set. Thus $l(e) = 0$, $l(s_k) = 1$, $l(s_k s_j) = 2$, and there is a unique maximal length element $w_0 \in W$ which is also a reflection. Hasse diagrams are graphs where vertices corresponding to elements of a poset are connected if $u > v$ and there is no intermediate w such that $u > w > v$. Figure 2 illustrates the Hasse diagrams of low-rank Weyl groups under the Bruhat order.

PROPOSITION 2.10 (Proposition 2.2.7 (Lifting Property) [**7**]). *Suppose $u < w$ and $s_i w < w$ but $s_i u \not< u$. Then, $u \leq s_i w$ and $s_i u \leq w$.*

PROPOSITION 2.11 (Proposition 2.3.1 [**7**]). (1) *If W is finite, there exists an element $w_0 \in W$ such that $w \leq w_0$ for all $w \in W$.*

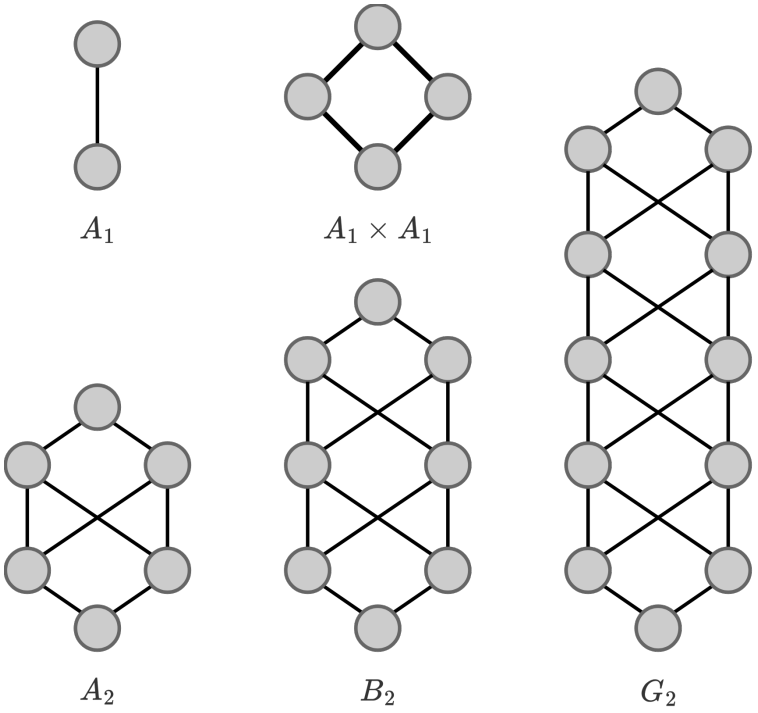

FIGURE 2. The Hasse diagrams of the Weyl groups of A_1, $A_1 \times A_1$, A_2, B_2, and G_2.

(2) *Conversely, suppose that* (W, Δ) *has an element* u *such that* $s_i u \leq u$ *for all* $s_i \in \Delta$. *Then,* W *is finite and* $x = w_0$.

PROPOSITION 2.12 (Proposition 2.3.2 [**7**]). *The unique longest element* w_0 *of a finite group has the following properties:*

(1) $w_0^2 = e$.

(2) $l(ww_0) = l(w_0) - l(w)$, *for all* $w \in W$.

(3) $s_k w < w$ *implies* $s_k w w_0 > w w_0$, *for all* $w \in W$.

3. Flag varieties, Schubert varieties, and Richardson varieties

We call $\mathcal{F} = G/B$ (or equivalently $K/(K \cap B)$) the *(full) flag variety* of G, and we call $\mathcal{F}_S = G/P_S$ the *S-flag variety* of G. Both are complex projective varieties, and if $S < S'$ then there are natural projections

$$\mathcal{F} \cong G/B \to G/P_{S'} \to G/P_S \to G/G \cong \{pt\}. \tag{42}$$

Thus G acts transitively on G/P_S.

We call B, B' *transverse* if their intersection is a maximal torus. Two parabolic subgroups P, P' are *incident* if their intersection contains a Borel subgroup and *transverse* if their intersection is a maximal Levi factor in each. Let $S \subset \Delta$. We call $S^* \subset \Delta$ the subset such that for some $g \in G$, P_S and $gP_{S^*}g^{-1}$ are transverse. If $S = S^*$, then we call S *symmetric*. We assume that $S \subset \Delta$ are symmetric.

Consider the flag $x = eB \in G/B$. We observe that B is the intersection of the standard maximal parabolic subgroups P_r where $r \in \Delta$, i.e.

$$B = (P_{r_1}, \ldots, P_{r_n}) \tag{43}$$

$$B = \bigcap_{i=1}^{n} P_{r_i}. \tag{44}$$

Using an explicit realization of the Tits geometry which identifies the standard maximal parabolic groups with subspaces in some $\mathbb{C}^n$, we may describe x as an n-tuple of incident subspaces:

$$x = (x_1, \ldots, x_n) \tag{45}$$

where $x_i \subset \mathbb{C}^N$ is stabilized by any $g_x \in B$. We may identify an arbitrary flag $y \in \mathcal{F}$ using a coset $g_y B \in G/B$, and describe y as a tuple of incident subspaces:

$$y = (g_y \cdot x_1, \ldots, g_y \cdot x_n) \tag{46}$$

$$= (y_1, \ldots, y_n) \tag{47}$$

Then the Borel subgroup $g_y B g_y^{-1}$ stabilizes each $y_i \subset \mathbb{C}^n$. We call this incidence description of the flag variety the *linear model of* G (see e.g. [**13**] for a detailed description). When G is simple, incidence between subspaces typically means containment, e.g. in Type A_n, flags are full flags of $\mathbb{C}^{n+1}$ such that x_i has dimension i and $x_i \subset x_{i+1}$. However, in Type D_n and E_n, incidence can mean that $x_i \cap x_j$ has a certain dimension [**13**]. Incidence between subspaces corresponding to different simple factors is trivial. Then elements of G/P_S are partial flags,

i.e. $|S|$-tuples where each $r \in S$ corresponds to a subspace x_r stabilized by (a conjugate of) P_S.

We call an element $g \in G$ *proximal* if $r(\lambda(g)) > 0$ for all $r \in \Delta$, or equivalently if there is a fixed point in $g^+ \in \mathcal{F}$ such that an open neighborhood of g^+ in $\mathcal{F}$ is attracted to g^+. Similarly, we call an element $g \in G$ *S-proximal* if $r(\lambda(g)) > 0$ for all $r \in S$, or equivalently if there is a fixed point $g^+ \in G/P_S$ such that an open neighborhood of g^+ in $\mathcal{F}_S$ is attracted to g^+. If g is S-proximal, then g^{-1} is S^*-proximal with fixed point $g^- \in G/P_{S^*}$ [**20**].

It is known that the subset of $\mathcal{F}$ (or resp. of $\mathcal{F}_S$) attracted to g^+ is the subset of $\mathcal{F}$ (resp. of $\mathcal{F}_S$) that is transverse to g^-. (e.g. Lemma 2.26 in [**20**]). In Chapter 3, we will describe the dynamics of proximal elements acting on $\mathcal{F}$.

3.1. Schubert varieties and Richardson varieties. Let $B < G$ be a Borel subgroup and $H < B$ a Cartan subgroup. Recall $W = N_G(H)/H$. While W is not in general a subgroup of G, it has a natural action on G and on $\mathcal{F}$. In particular, it acts on B which is stabilized by H, e.g. $eB = B$ and $w_0 B = B^-$ is the transverse Borel subgroup such that $B \cap B^- = H$.

For each $w \in W$, we may consider the double coset BwB in $\mathcal{F}$, called an *open Schubert cell*. Then BwB is a complex affine subvariety of $\mathcal{F}$ homeomorphic to $\mathbb{C}^{l(w)}$ [**10**]. The Schubert cells partition $\mathcal{F}$, i.e.

$$\mathcal{F} = \bigsqcup_{w \in W} BwB. \tag{48}$$

This is called the Bruhat decomposition of $\mathcal{F}$ [**3**]. The closure of the open Schubert cells are the *Schubert varieties*, which are complex projective subvarieties of $\mathcal{F}$ [**10**]. In particular,

$$\overline{BwB} = \bigsqcup_{u \leq w} BuB. \tag{49}$$

We describe $\overline{BwB}$ in the special case where $w = s_k$ explicitly in our linear model of G. Recall that $x = eB \in G/B$. Then

$$x = (x_1, \dots, x_n), \tag{50}$$

and the Schubert variety consists of flags where the i-th subspace is x_i for any $i \neq k$, and the k-th subspace is any translate $\widetilde{x_k} = g \cdot x_k$ which is incident to x_i for $i \neq k$, i.e.

$$\overline{Bs_kB} = \{(x_1, \dots, x_{k-1}, \widetilde{x_k}, x_{k+1}, \dots, x_n) : \widetilde{x_k} \text{ incident to } x_i\}. \tag{51}$$

Since $Bs_kB \cong \mathbb{C}$, thus $\overline{Bs_kB} \cong \mathbb{CP}^1$ for all $s_k \in \Delta \subset W$ [**10**].

Recall that B^- is the standard Borel subgroup transverse to B, i.e. w_0B. The *Richardson varieties* are nonempty intersections of the form

$$\overline{BwB} \cap \overline{B^-uB^-}, \tag{52}$$

where $B^-uB^- = w_0Buw_0B$. In particular, if $u \leq w$, they are connected complex projective subvarieties of dimension $l(w) - l(u)$, and if $u \not\leq w$ then the intersection is empty [**33**]. There is also a Richardson decomposition of $\mathcal{F}$, i.e.

$$\mathcal{F} = \bigsqcup_{u,v \in W} BwB \cap B^-uB^-. \tag{53}$$

We will introduce our own notation for Schubert and Richardson varieties in Chapter 3.

4. Anosov and hyperconvex representations

Let X and Y be topological spaces equipped with a continuous action by a group Γ.

Definition 2.13 (Topological conjugacy [**31**]). Suppose $F : X \to Y$ is a homeomorphism which if Γ-equivariant, i.e. such that for all $x \in X$ and $\gamma \in \Gamma$ we have $\gamma \cdot F(x) = F(\gamma \cdot x)$. We call such F a *topological conjugacy*.

In Chapter 5, we discuss topological conjugacies between Anosov representations.

4.1. Anosov representations. We focus on Anosov representations in complex semisimple Lie groups, though they may be defined in reductive Lie groups over other fields [**20**].

Let Γ be a word hyperbolic group with boundary $\partial\Gamma$. Let G be a semisimple Lie group of rank $n \geq 1$. Let $S \subset \Delta$ be a symmetric subset of the positive simple roots, and $P_S < G$ be the associated parabolic. Then $\mathcal{F}_S = G/P_S$.

Definition 2.14 (S-Anosov representation [**20**]). Let $\rho : \Gamma \to G$ be a representation. Suppose there exists a *continuous, ρ-equivariant boundary map* $\xi : \partial\Gamma \to \mathcal{F}_S$ with the following properties:

(1) ξ is *transverse*, i.e. if $x \neq y \in \partial\Gamma$, then $\xi(x)$ and $\xi(y)$ are transverse in $\mathcal{F}_S$.
(2) ξ is *dynamics-preserving*, i.e. if $\gamma \in \Gamma$ infinite order, then $\rho(\gamma)$ is proximal with attracting fixed point $\xi(\gamma^+)$ and repelling fixed point $\xi(\gamma^-)$.
(3) ρ is *S-expanding*, i.e. for any $r \in S$ and sequence $\gamma \in \Gamma^{\mathbb{N}}$ such that the word length diverges we have

$$\limsup r(\mu(\rho(\gamma))) = +\infty. \tag{54}$$

Here μ is the Cartan projection, and $r \in \mathfrak{a}^*$ is a real functional.

We call such representations *S-Anosov*.

In this thesis, we do not prove that representations are Anosov, but instead start with an Anosov representation and use the existence of its continuous, ρ-equivariant, transverse, dynamics-preserving boundary map ξ. We do not engage further with the requirement that ρ is S-expanding. The boundary map ξ of an Anosov representation is unique, and we call the image of ξ the *limit set* $\Lambda \subset \mathcal{F}$.

If $S = \Delta$ or equivalently $P_S = B$ the Borel subgroup, then we call ρ Borel Anosov, and the boundary map $\xi : \partial\Gamma \to \mathcal{F} \cong G/B$. In Chapter 5, we assume that ρ is Borel Anosov. If ρ is S-Anosov and $T \subset S$ is another subset of Δ, then ρ is T-Anosov [**20**].

Anosov representations have several desireable properties. In particular, they are discrete quasi-isometric embeddings of Γ and have finite kernel. The subset of S-Anosov representations $An_S(\gamma, G) \subset \mathrm{Hom}(\Gamma, G)$ is open [**29**]. The limit set varies continuously with respect to the space of representations [**23**]. Additionally, if ρ is S-Anosov, then for any infinite-order $\gamma \in \Gamma$ then $\rho(\gamma)$ is proximal in G/P_S [**20**].

4.2. Hyperconvex representations. Here we specify that $G = SL_{n+1}\mathbb{C}$. Then $W \cong \mathrm{Sym}(n+1)$, and we may identify s_k with the transposition $(k, k+1) \in \Delta \subset W$ for $k \in \{1, \dots, n\}$. If $\rho : \Gamma \to SL_{n+1}\mathbb{C}$ is k-Anosov then it is also $(n+1-k)$-Anosov, since $S^* = \{n+1-k : k \in S\}$.

Definition 2.15 (Hyperconvex). Let $S_k = \{k-1, d-k, k+1\}$. Let $\rho : \Gamma \to SL_{n+1}\mathbb{C}$ be an S_k-Anosov representation with limit set $\Lambda \subset \mathcal{F}_{S_k}$. Suppose for all distinct $x, y, z \in \Lambda$ we have

$$(x_{k-1} + y_{d-k} \cap x_{k+1}) \cap (x_{k-1} + z_{d-k} \cap x_{k+1}) = x_{k-1}. \tag{55}$$

We call such representations *k-hyperconvex*, or *hyperconvex* if they are k-hyperconvex for all $1 \leq k \leq n$.

We observe that the set of k-hyperconvex representations is a subset of the set of S_k-Anosov representations, i.e.

$$\mathrm{Hyp}_k(\Gamma, SL_{n+1}\mathbb{C}) \subset An_{S_k}(\Gamma, SL_{n+1}\mathbb{C}) \subset \mathrm{Hom}(\Gamma, SL_{n+1}\mathbb{C}). \tag{56}$$

Similarly, hyperconvex representations are Borel Anosov.

Hyperconvex representations admit continuous boundary projections to $\mathbb{CP}^1$. Let ρ be k-hyperconvex and let $x \in \Lambda$. We may define the projection

$$\pi_k^x : \Lambda \to \mathbb{P}(x_{k+1}/x_{k-1}) \cong \mathbb{CP}^1 \tag{57}$$

$$\pi_k^x(y) = \begin{cases} y_{d-k} \cap x_{k+1} : y \neq x \\ x_k : y = x \end{cases} . \tag{58}$$

Clearly π_k^x is well-defined and continuous on $\Lambda \setminus \{x\}$ for any S_k-Anosov representation, and injective because ρ is k-hyperconvex. It was shown that π_k^x is continuous on (all of) Λ in [**6**].

Recent results in the literature have applied classical quasiconformal theory to hyperconvex representations. For $\Gamma = \pi_1\Sigma$, Farre, Pozzetti, and Viaggi parameterize $\mathrm{Hyp}(\pi_1\Sigma, SL_{n+1}\mathbb{C})$ using a natural holomorphic extension of the Ahlfors-Bers parameters to a product of the Teichmuller space of surface laminations [**18**]. They also show that any Γ admitting a hyperconvex representation is virtually isomorphic to a Kleinian group [**19**].

Hyperconvex representations have not been largely considered beyond the context of $G = SL_{n+1}\mathbb{R}$ or $G = SL_{n+1}\mathbb{C}$. However, using a different definition of hyperconvexity, Sambarino considered hyperconvex representations in $PSL_2\mathbb{C} \times PSL_2\mathbb{C}$ [**34**]. Additionally, Davalo used hyperconvexity (in the sense of $SL_4\mathbb{R}$) to prove that maximal representations in $Sp_4\mathbb{R}$ are contained in the $SL_4\mathbb{R}$-Hitchin component [**15**]. The definition for hyperconvexity proposed in Chapter 4 allows us to consider hyperconvexity with respect to the flag variety of a semisimple Lie group G, rather than as a partial flag variety of some $SL_N\mathbb{C}$.

CHAPTER 3

Richardson varieties describe the dynamics of proximal elements on $\mathcal{F}$

The purpose of this chapter is to describe Richardson varieties in the full flag variety of a complex semisimple Lie group, and use them to describe the dynamics of the action of proximal elements on the flag variety. The results in this section are likely already known to those who study flag varieties.

1. Notation

Let G be a complex semisimple Lie group of rank $n \geq 2$. Let $B < G$ be a minimal parabolic subgroup, and let $\mathcal{F} = G/B$ be the flag variety.

EXAMPLE 3.1. *In the linear model of $SL_{n+1}\mathbb{C}$, we may view $x \in \mathcal{F}(\mathbb{C}^n)$ as an n-tuple*

$$x = (x_1, \ldots, x_n) \tag{59}$$

where each x_k is a k-dimensional linear subspace of $\mathbb{C}^{n+1}$ and $x_k \subset x_{k+1}$.

In the linear model of semisimple G, we may identify $x \in \mathcal{F}$ as an n-tuple of incident linear subspaces of a complex vector space, so that each x_k corresponds to a maximal parabolic P_{r_k}, as discussed in Chapter 2, Section 3. For each $1 \leq k \leq n$, the subspace x_k may not have dimension k, and may be subject to additional conditions e.g. the vanishing of a bilinear form. We describe the incidence geometry of other complex simple Lie groups in Chapter 4, Section 7.

Let W be the Weyl group of G. Then W is a finite Coxeter group, and let $\{s_i\}_{i=1}^n \in W$ be the generating reflections. Each s_k corresponds to a simple root $r_k \in \Delta$.

EXAMPLE 3.2. *When $G = SL_{n+1}\mathbb{C}$, then $W \cong Sym(n+1)$, the symmetric group on $n+1$ letters. Under this isomorphism we may take each s_k to be the adjacent transposition $(k, k+1)$.*

Let l refer to the Bruhat length of an element, i.e. the length of a minimal word for w using the generating reflections. Then $\dim(\alpha_w(x)) = l(w)$. Let $\geq$ refer to the Bruhat order on W, and $\geq_L$ refer to the weak left Bruhat order. Recall that $u \geq v$ in the Bruhat order if

and only if $\alpha_u(x) \supset \alpha_v(x)$, and $u \geq_L v$ if there exists a sequence of generators $\{s_{k_i}\}_{i=1}^N$ such that

$$u = s_{k_N} \cdots s_{k_1} v > \cdots > s_{k_1} v > v \tag{60}$$

We may identify a flag $x \in \mathcal{F}$ with the coset $gB \in G/B$. Let $w \in W$, define the *principal w-Schubert variety* based at x as follows:

$$\alpha_w(x) = g\overline{BwB} = g(\bigsqcup_{u \leq w} BwB). \tag{61}$$

Then $\alpha_w(x)$ is an irreducible complex projective variety containing x. For any generator s_k, we use the notation $\alpha_k(x) := \alpha_{s_k}(x)$.

EXAMPLE 3.3. *In the linear model of $SL_{n+1}\mathbb{C}$, the principal s_k-Schubert variety based at $x \in \mathcal{F}(\mathbb{C}^{n+1})$ is*

$$\alpha_k(x) := \{(x_1, \ldots, x_{k-1}, \widetilde{x_k}, x_{k+1}, \ldots, x_n) : x_{k-1} \subset \widetilde{x_k} \subset x_{k+1}\}. \tag{62}$$

As explained in Chapter 2, Section 3.1, since the subspace x_k corresponds to the root r_k which correspond to the generating reflection s_k, then $\alpha_k(x)$ consists of the elements of $\mathcal{F}$ differing from x in only the k-th subspace.

We may use W to describe the relative position of two flags, x and y. Recall that there are unique Bruhat-minimal elements $w(x,y), w(y,x) \in W$ such that $y \in \alpha_{w(x,y)}(x)$ and $x \in \alpha_{w(x,y)}(y)$. Observe that $w(x,y) = w(y,x)^{-1}$.

Recall $w_0 \in W$ the unique longest Bruhat-length element. Then $l(w_0) = \dim(\mathcal{F})$ and $\alpha_{w_0}(x) = \mathcal{F}$. We call x and y *transverse* if $w(x,y) = w(y,x) = w_0$. The condition $w(x,y) = w_0$ holds for an open, dense subset of pairs $(x,y) \in \mathcal{F} \times \mathcal{F}$.

EXAMPLE 3.4. *In the linear model of $SL_{n+1}\mathbb{C}$, pairs $x, y \in \mathcal{F}(\mathbb{C}^{n+1})$ are transverse if and only if the sum $x_k + y_{n+1-k}$ is direct for all $1 \leq k \leq n$.*

Using the standard basis $e_1, \dots, e_{n+1}$, let $x, y \in \mathcal{F}$ as follows:

$$x = (e_1, e_1 \oplus e_2, \dots, e_1 \oplus \cdots \oplus e_n) \tag{63}$$

$$y = (e_{n+1}, e_{n+1} \oplus e_n, \dots, e_{n+1} \oplus \cdots \oplus e_2). \tag{64}$$

Then x and y are transverse.

Richardson varieties are the nonempty intersections of principle Schubert varieties of pairs of transverse flags. Let $x, y \in \mathcal{F}$ be transverse. Then

$$\text{Rich}_{x,y} = \{\alpha_u(x) \cap \alpha_w(y) : w, u \in W\}. \tag{65}$$

We use $\text{Rich}^k_{x,y}$ to refer to the set of k-dimensional Richardson varieties (briefly, the Richardson k-varieties) of x and y.

2. Richardson varieties describe the dynamics of proximal elements on $\mathcal{F}$

We state some properties related to Richardson varieties. Let $x, y \in \mathcal{F}$ be transverse, and let $\iota : W \to W$ given by $w \mapsto ww_0$ so that $\iota(w)^{-1}w = w_0$ for all $w \in W$.

PROPOSITION 3.5. *Then there is a bijection between W and $Rich^0_{x,y}$ given by*

$$w \mapsto x^y_w := \alpha_w(x) \cap \alpha_{\iota(w)}(y). \tag{66}$$

If $u, v \in W$ such that $u < v$ and $l(u) + 1 = l(v)$ we call v an *upper cover* of u.

PROPOSITION 3.6. *There is a bijection between pairs $(u, v) \in W^2$ such that u is an upper cover of v and $Rich^1_{x,y}$ given by*

$$(u, v) \mapsto \alpha_u(x) \cap \alpha_{\iota(v)}(y). \tag{67}$$

Thus the Hasse diagram (as introduced in Chapter 2, Section 2.4) of W under the Bruhat order describes the Richardson 0- and 1-varieties, in that the vertices (e.g. w) are in bijection with the 0-varieties (e.g. x^y_w) and the edges (e.g. $u > v$) are in bijection with the 1-varieties (e.g. $\alpha_u(x) \cap \alpha_{\iota(v)}(y)$). An example is illustrated in Figure 1.

Richardson varieties vary continuously among transverse pairs.

PROPOSITION 3.7. *Suppose* $\{x(i)\}_{i=1}^{\infty}, \{y(i)\}_{i=1}^{\infty} \subset \mathcal{F}$ *such that*

- *the pair* $x(i), y(i)$ *is transverse for all* $i \in \mathbb{N}$,
- $\lim_{i\to\infty} x(i) = x$ *and* $\lim_{i\to\infty} y(i) = y$, *and*
- x *and* y *are transverse.*

Then for all $w \in W$ *we have*

$$\lim_{i\to\infty} x(i)_w^{y(i)} = x_w^y. \tag{68}$$

The proof is provided in Section 4.

Richardson varieties describe the dynamics of proximal elements on the flag variety.

THEOREM 3.8. *Let* $\gamma \in G$ *be proximal, and define* $x := \gamma^+$ *and* $y := \gamma^-$.

Let $p \in \mathcal{F}$, *and define* $u := w(x, p)$ *and* $v := w(y, p)$ *the relative positions of* x *with* p *and* y *with* p. *Then*

$$\lim_{i\to\infty} \gamma^i \cdot p = y_v^x, \text{ and} \tag{69}$$

$$\lim_{i\to\infty} \gamma^{-i} \cdot p = x_u^y. \tag{70}$$

The proof is provided in Section 5. We also highlight the fixed points of $\mathcal{F}$ under the action of γ:

COROLLARY 3.9. *The fixed point set of* γ *acting on* $\mathcal{F}$ *is* $Rich_{x,y}^0$, *where* $x = \gamma^+$ *and* $y = \gamma^-$.

3. Properties of Richardson varieties

Throughout this section, assume that $x, y \in \mathcal{F}$ are transverse. Recall the order-reversing Weyl group involution $\iota : W \to W$ which sends w to ww_0.

Richardson proved the following in his paper introducing Richardson varieties [**33**]. The original notation differs significantly from ours.

PROPOSITION 3.10. *Let $u, v \in W$.*

- *If $\iota(v) \leq u$, then $\alpha_u(x) \cap \alpha_v(y)$ is an irreducible complex projective variety of dimension $l(u) + l(v) - l(w_0)$.*
- *Otherwise, the Schubert varieties $\alpha_u(x)$ and $\alpha_v(y)$ are disjoint.*

Propositions 3.5 and 3.6 may be seen as special cases of Proposition 3.10. Since Richardson 0-varieties are in bijection with the Weyl group, we use the notation

$$x_w^y := \alpha_w(x) \cap \alpha_{\iota(w)}(y). \tag{71}$$

Given a pair of transverse flags in the linear model of G, the Richardson 0-varieties may be computed explicitly.

EXAMPLE 3.11. *Consider the linear model of the flag variety of $SL_3\mathbb{C}$. Then the Richardson 0-varieties of transverse $x, y \in \mathcal{F}(\mathbb{C}^3)$ are as follows:*

- $x_e^y = (x_1, x_2)$,
- $x_{s_1}^y = (x_2 \cap y_2, x_2)$,
- $x_{s_2}^y = (x_1, x_1 + y_1)$,
- $x_{s_2 s_1}^y = (x_2 \cap y_2, y_2)$,
- $x_{s_1 s_2}^y = (y_1, x_1 + y_1)$, *and*
- $x_{w_0}^y = (y_1, y_2)$.

In Chapter 4, Section 7, we compute Richardson 0-varieties in other simple Lie groups.

REMARK 3.12. *The notation highlights the relative position of Richardson 0-varieties. In particular,*

- $w(x, x_w^y) = w$,
- $x_w^y = y_{\iota(w)}^x$,
- $x_e^y = x$,
- $x_{w_0}^y = y$, *and*
- x_w^y *and* $x_{w_0 w}^y$ *are transverse.*

Next, we describe families of embedded copies of $\mathbb{CP}^1$ in $\mathcal{F}$. We call distinct flags $p, q \in \mathcal{F}$ k-adjacent if $p \in \alpha_k(q)$. Using our description of a flag as an n-tuple of incident subspaces, two flags are k-adjacent if they differ only in the k-th subspace, which may or may not have dimension k. We define the span of such p and q to be the s_k-Schubert variety based at either p or q, i.e.

$$\text{span}(p, q) := \alpha_k(p) = \alpha_k(q). \tag{72}$$

The Richardson 1-varieties of x and y are in bijection with pairs $u, v \in W$ such that u is an upper cover of v, i.e. $u > v$ and $l(u) = l(v) + 1$. If for some $1 \leq k \leq n$ we have $u = s_k v$, then $u >_L v$ and u is an *left upper cover* of v. In this case, x_u^y and x_v^y are k-adjacent, and the Richardson 1-variety

$$\alpha_u(x) \cap \alpha_{\iota(v)}(y) = \alpha_k(x_u^y) = \alpha_k(x_v^y). \tag{73}$$

If u is an upper cover, but not a left upper cover of v, then x_u^y and x_v^y are not k-adjacent for any k. We define the *span* of x_u^y and x_v^y to be the Richardson 1-variety containing both, and its *type* is given by the relative position of x_u^y and x_v^y in $\mathcal{F}$. In particular, the relative position is a reflection in W, since $w(x_u^y, x_v^y) = w(x_v^y, x_u^y)$.

PROPOSITION 3.13. *Let $p, q, p', q' \in \mathcal{F}$ and consider the rational curve span(p, q) and span$(p', q') \subset \mathcal{F}$. Then one of the following is true:*

(1) *span(p, q) = span(p', q') and they are of the same type,*
(2) *span$(p, q) \cap$ span$(p', q') = \{pt\}$ and if span$(p, q) = \alpha_k(p)$ and span$(p', q') = \alpha_j(p')$ then $k \neq j$,*
(3) *span(p, q) and span(p', q') are disjoint.*

PROOF. A span is determined by any pair of distinct points in it, and all distinct elements of span(p, q) have the same relative position as p and q. Thus if span$(p, q) \cap$ span(p', q') has cardinality greater than one, the sets are equal. □

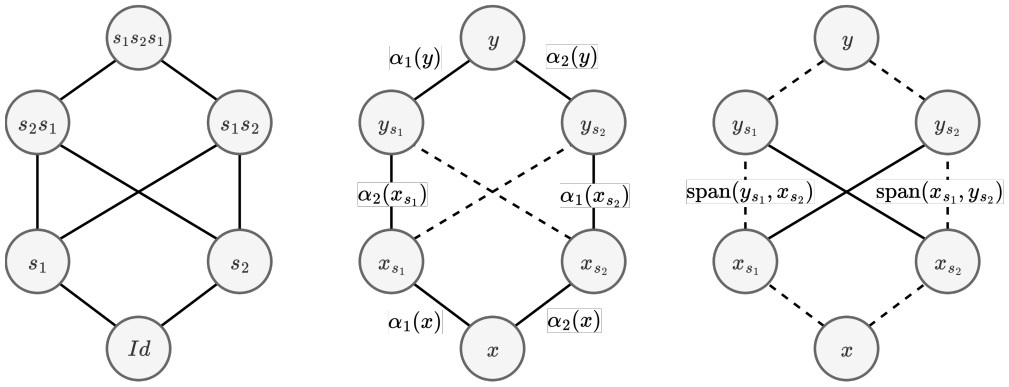

FIGURE 1. The Hasse diagram of $W(A_2)$ encodes the 6 Richardson 0-varieties and 8 Richardson 1-varieties of transverse $x, y \in \mathcal{F}(\mathbb{C}^3)$.

Thus the Richardson 0- and 1-varieties may be described using the Hasse diagram of the Weyl group, where vertices are 0-varieties and edges are 1-varieties. Figure 1 illustrates this concept: the first panel is the Hasse diagram of A_2, the second labels the 6 Richardson 0-varieties and 6 k-span Richardson 1-varieties, and the third labels the remaining Richardson 0-varieties where

$$\mathrm{span}(x^y_{s_2}, y^x_{s_1}) = \{(\widetilde{x_1}, \widetilde{x_1} + y_1) \in \mathcal{F}(\mathbb{C}^3) : \widetilde{x_1} \in x_2\}, \mathrm{span}(x^y_{s_1}, y^x_{s_2}) \quad = \{(\widetilde{y_1}, \widetilde{y_1} + x_1) \in \mathcal{F}(\mathbb{C}^3) : \widetilde{y_1} \in y_2\}. \tag{74}$$

EXAMPLE 3.14. *Consider the linear model of $SL_3\mathbb{C}$. Let $x, y \in \mathcal{F}(\mathbb{C}^3)$ be transverse. Then there are eight Richardson 1-varieties of x and y, several illustrated in Fig. 2.*

Six are the span of k-adjacent pairs:

- $span(x, x^y_{s_1}) = \alpha_1(x)$,
- $span(x, x^y_{s_2}) = \alpha_2(x)$,
- $span(x^y_{s_1}, y^x_{s_1}) = \alpha_2(x^y_{s_1})$,
- $span(x^y_{s_2}, y^x_{s_2}) = \alpha_1(x^y_{s_2})$,
- $span(y, y^x_{s_1}) = \alpha_1(y)$, *and*
- $span(y, y^x_{s_2}) = \alpha_2(y)$.

Two are the span of non-adjacent pairs, described explicitly:

- $span(x^y_{s_2}, y^x_{s_1}) = \{(\widetilde{x_1}, \widetilde{x_1} + y_1) \in \mathcal{F}(\mathbb{C}^3) : \widetilde{x_1} \in x_2\}$, *and*

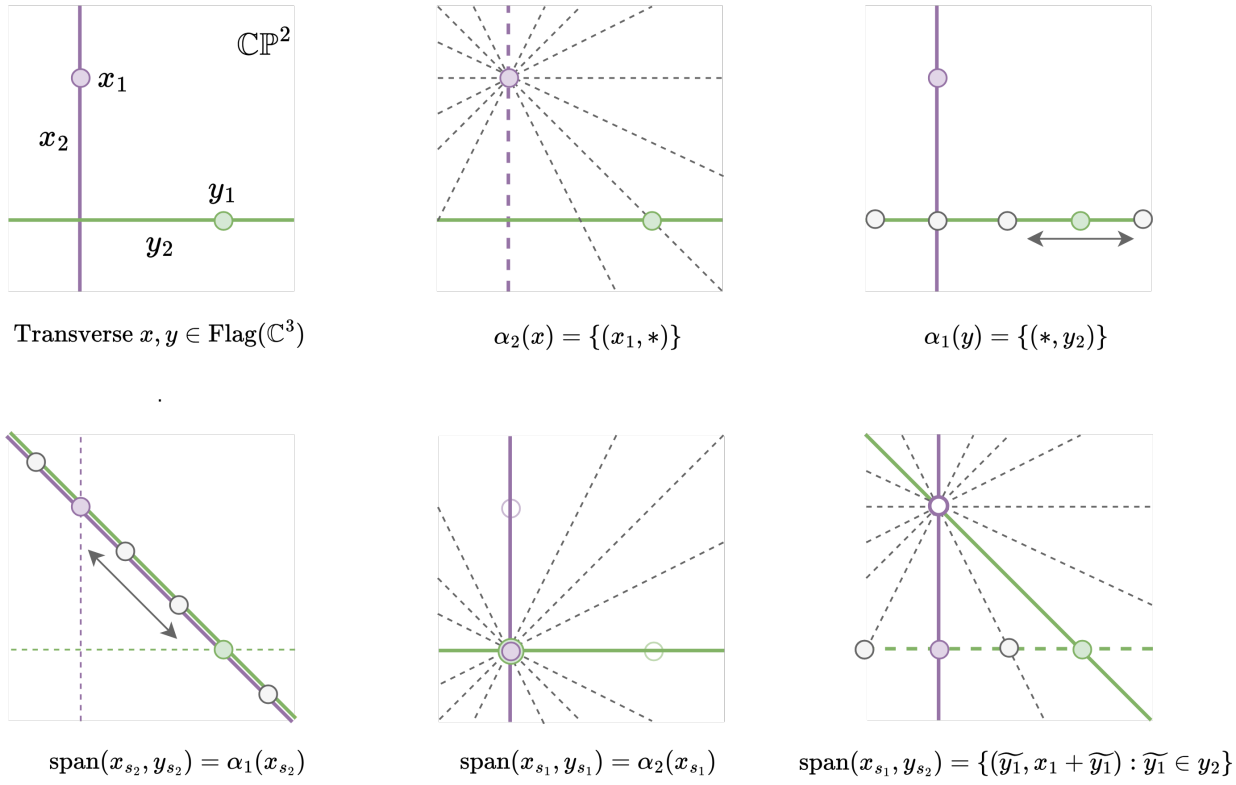

FIGURE 2. Projective model for $\mathcal{F}(\mathbb{C}^3)$ (top left) and selected Richardson 1-varieties in $\mathcal{F}(\mathbb{C}^3)$.

- $span(x^y_{s_1}, y^x_{s_2}) = \{(\widetilde{y_1}, \widetilde{y_1} + x_1) \in \mathcal{F}(\mathbb{C}^3) : \widetilde{y_1} \in y_2\}$.

4. Proof of Proposition 3.7

We demonstrate that Richardson varieties vary continuously on transverse pairs of flags.

PROOF OF PROPOSITION 3.7. Let $\{x(i)\}_{i=1}^{\infty}, \{y(i)\}_{i=1}^{\infty} \subset \mathcal{F}$ and $x, y \in \mathcal{F}$ be as assumed.

Let $w \in W$. Observe that a maximal compact subgroup $K < G$ acts transitively on $\mathcal{F}$. So there are sequences $\{j(i)\}_{i=1}^{\infty}, \{k(i)\}_{i=1}^{\infty} \in K^{\mathbb{N}}$ such that $\lim_{i\to\infty} j(i) = \lim_{i\to\infty} k(i) = e$, $j(i) \cdot x(i) = x$ for all $i \in \mathbb{N}$, and $k(i) \cdot y(i) = y$ for all $i \in \mathbb{N}$. Define $y'(i) = j(i) \cdot y(i)$ and $x'(i) = k(i) \cdot x(i)$. Clearly the pairs $x, y'(i)$ and $x'(i), y$ are transverse for all i, since action by an element of G does not change the relative position of two flags. By construction, the sequences $x(i)^{y(i)}_w$, $x^{y'(i)}_w = j(i) \cdot (x(i)^{y(i)}_w)$, and $x'(i)^y_w = k(i) \cdot (x(i)^{y(i)}_w)$ have the same limiting

behavior, meaning that any limit point for one is a limit point for the others, since $j(i)$ and $k(i)$ converge to the identity in G.

Clearly $x_w^{y'(i)} \in \alpha_w(x)$ for all i, and $\alpha_w(x)$ is compact, so $x_w^{y'(i)}$ has a convergent subsequence with limit $z_x \in \alpha_w(x)$. Similarly, $x'(i)_w^y \in \alpha_{\iota w}(y)$ for all i, so $x'(i)_w^y$ has a convergent subsequence with limit $z_y \in \alpha_{\iota(w)}(y)$. But since the sequences have the same limiting behavior, then z_x is a limit point of $x'(i)_w^y$, and z_y is a limit point of $x_w^{y'(i)}$, so in particular $z_x \in \alpha_{\iota(w)}(y)$ and $z_y \in \alpha_w(x)$. Since x and y are transverse, then the intersection $\alpha_w(x) \cap \alpha_{\iota(w)}(y)$ is a singleton, so $z_x = z_y = x_w^y$. Thus every convergent subsequence of $x_w^{y'(i)}$ and $x'(i)_w^y$ limits to x_w^y. Since $\mathcal{F}$ is compact and Hausdorff, this shows the limit of each sequence exists and is x_w^y.

Since all three sequences have the same limiting behavior, we have

$$\lim_{i\to\infty} x(i)_w^{y(i)} = \lim_{i\to\infty} x_w^{y'(i)} \tag{75}$$

$$= \lim_{i\to\infty} x'(i)_w^y \tag{76}$$

$$= x_w^y. \tag{77}$$

Since w was arbitrary, then we are done. □

Since $\mathrm{Rich}^0_{x,y}$ has fixed cardinality $|W|$, we abuse notation to say that the sets of Richardson 0-varieties converge:

$$\lim_{i\to\infty} \mathrm{Rich}^0_{x(i),y(i)} = \{\lim_{i\to\infty} x(i)_w^{y(i)} : w \in W\} \tag{78}$$

$$= \{x_w^y : w \in W\} \tag{79}$$

$$= \mathrm{Rich}^0_{x,y}. \tag{80}$$

Similarly, since $w(x(i), x(i)_w^{y(i)}) = w(x, x_w^y) = w$ and $w(y(i), y(i)_u^{x(i)}) = w(y, y_u^x) = u$, we may further abuse notation to say that

$$\lim_{i\to\infty} \alpha_w(x(i)) \cap \alpha_u(y(i)) = \alpha_w(\lim_{i\to\infty} x(i)) \cap \alpha_u(\lim_{i\to\infty} y(i)) \tag{81}$$

$$= \alpha_w(x) \cap \alpha_u(y). \tag{82}$$

In particular, for all $p \in \alpha_w(x)$, there is a sequence $p(i) \in \alpha_w(x(i))$ such that $\lim_{i\to\infty} p(i) = p$.

5. Proof of Theorem 3.8

Let $\gamma \in G$ be a proximal element. Then there are $x, y \in \mathcal{F}$ which are attracting and repelling fixed points of the action of γ on $\mathcal{F}$. Specifically, there is a neighborhood U_γ of x in $\mathcal{F}$ such that for all $p \in U_\gamma$, we have

$$\lim_{i\to\infty} \rho(\gamma^i) \cdot p = x. \tag{83}$$

Correspondingly, there is a neighborhood V_γ of y in $\mathcal{F}$ such that for all $p \in V_\gamma$, we have

$$\lim_{i\to\infty} \rho(\gamma^{-i}) \cdot p = y. \tag{84}$$

While there is an open set of $\mathcal{F}$ attracted to x and repelled by y, there is also an exceptional locus for the action of γ consisting of flags attracted to a point other than x or repelled by a point other than y.

For example, the Richardson 0-varieties are exceptional because they are fixed points.

Proposition 3.15. *The Richardson 0-varieties of $x = \gamma^+$ and $y = \gamma^-$ are fixed points of $\gamma \cdot \mathcal{F}$.*

Proof. The relative position of any two flags p and q is the same as that of $\gamma \cdot p$ and $\gamma \cdot q$, i.e. $w(p, q) = w(\gamma \cdot p, \gamma \cdot q)$. Then for all $u, v \in W$, we have $\gamma \cdot \alpha_u(x) = \alpha_u(x)$ and

$\gamma \cdot \alpha_v(y) = \alpha_v(y)$, so

$$\gamma \cdot (\alpha_u(x) \cap \alpha_v(y)) = \gamma \cdot (\alpha_u(x)) \cap \gamma \cdot (\alpha_v(y)) \tag{85}$$

$$= \alpha_u(x) \cap \alpha_v(y). \tag{86}$$

In particular, the Richardson varieties of x and y are invariant under the action of γ. Let $w \in W$, and choose $v = \iota(w)$. Then we have

$$\gamma \cdot \{x_w^y\} = \gamma \cdot (\alpha_w(x) \cap \alpha_{\iota(w)}(y)) \tag{87}$$

$$= \alpha_w(x) \cap \alpha_{\iota(w)}(y) \tag{88}$$

$$= \{x_w^y\}. \tag{89}$$

Thus γ fixes the Richardson 0-varieties of x and y. □

As an alternate argument, we could observe that $\gamma, \gamma^{-1} \in G$ are contained in some Cartan subgroup $H < gB < G$, then $N_G(H)gB$ is finite in G/B, and all such elements are stabilized by H. In particular, the action of H on $\mathcal{F}$ fixes $\{x_w^y\}_{w \in W} \subset \mathcal{F}$.

Next, we describe the dynamics of γ acting on Richardson 1-varieties containing k-adjacent Richardson 0-varieties. These are also exceptional.

PROPOSITION 3.16. *Let $u, v \in W$ such that u is a left upper cover of v. Then $u = s_k v$ for some $1 \leq k \leq n$, and γ acts loxodromically on span$(x_u^y, x_v^y) = \alpha_k(x_u^y)$ with attracting fixed point x_v^y and repelling fixed point x_u^y.*

PROOF. Observe that the action of γ restricted to a Richardson 1-variety of x and y is a linear action on $\alpha_k(x_u^y) \cong \mathbb{CP}^1$, and therefore either trivial (all points fixed), elliptic (rotation around two fixed points), parabolic (rotation around one fixed point), or loxodromic (distinct attracting and repelling fixed points). Clearly $x_u^y, x_v^y \in \alpha_k(x_u^y)$ are distinct fixed points, so the action is not parabolic.

Case 1: suppose that $v = e$ and $u = s_k$. Then $U_\gamma \cap \alpha_k(x)$ is a neighborhood of x in $\alpha_k(x)$ attracted to x, so the action of γ has an attracting fixed point and is therefore loxodromic.

Then there is a unique repelling fixed point, which is $x^y_{s_k}$. So p is attracted to x and repelled by $x^y_{s_k}$.

Case 2: suppose that $u = w_0$ and $v = s_k w_0$. Then $V_\gamma \cap \alpha_k(y)$ is a neighborhood of y in $\alpha_k(y)$ repelled by y, so the action of γ has a repelling fixed point and is therefore loxodromic. So there is a unique attracting fixed point, which is $y^x_{s_k}$. So p is attracted to $y^x_{s_k} = x^y_v$ and repelled by $y = x^y_u$.

Suppose $u >_L v$ and $u = s_k v$, and consider the action of γ on $\alpha_k(x^y_v)$. Consider the Lyapunov projection $\lambda_\gamma = \lambda(\gamma) \in \overline{\mathfrak{a}^+}$. There is a functional $r \in \overline{\mathfrak{a}^+}^*$ such that $r(\lambda_\gamma) > 0$ implies that γ acts loxodromically with attracting fixed point x^y_v, $r(\lambda_\gamma) < 0$ implies that γ acts loxodromically with attracting fixed point x^y_u, and $r(\lambda_\gamma) = 0$ means that γ acts elliptically.

Consider the tangent space of the flag variety at x denoted $Tx\mathcal{F}$, which is isomorphic to $galg/\mathfrak{b}$. Using our Cartan decomposition of $\mathfrak{g}$, then

$$Tx\mathcal{F} \cong \bigoplus_{r \in \mathcal{R}^-} \mathfrak{g}_r. \tag{90}$$

The tangent space of $\alpha_k(x)$ (denoted $T_x\alpha_k(x)$) is isomorphic to $\mathfrak{g}_{-r_k}$. Consider the Lyapunov projection of γ $\lambda_\gamma = \lambda(\gamma) \in \overline{\mathfrak{a}^+}$. Since γ fixes x and stabilizes $\alpha_k(x)$, then the action of γ on $T_x\alpha_k(x)$ is described by $r_k(\lambda_\gamma)$. We observe that $r_k(\lambda_\gamma) > 0$ and thus γ attracts a neighborhood of $\alpha_k(x)$ to x.

Consider $u >_L v$ such that $u = s_k v$. We want to understand the action of γ on $T_{x^y_v}\alpha_k(x^y_v)$. Since $\mathcal{F}$ is a homogeneous space, then we may translate x to $v \cdot x = x^y_v$. Then the action of γ on $T_{x^y_v}\alpha_k(x^y_v)$ is described by the translation of r_k by v, i.e.

$$r_k(v \cdot \lambda_\gamma) = v \cdot r_k(\lambda_\gamma). \tag{91}$$

We recall that W acts on the roots $\mathcal{R}$ via reflection, and since γ is proximal then $w \cdot r_k(\lambda_\gamma) \neq 0$ for all $w \in W$. This means that the action of γ on $\alpha_k(x^y_v)$ is loxodromic, and in particular x^y_v either attracts or repels a neighborhood of $\alpha_k(x^y_v)$. Since $u = s_k v$ then $v \cdot r_k(\lambda_\gamma) = -u \cdot r_k(\lambda_\gamma)$,

and since $u > v$ then $v \cdot r_k(\lambda_\gamma) > 0$. So x_v^y is the attracting fixed point of the action of γ on $\alpha_k(x_v^y)$, and x_u^y is the repelling fixed point. □

We use these statements to prove the Theorem, which describes the dynamics of γ acting on $\mathcal{F}$.

PROOF OF THEOREM 3.8. Let $p \in \mathcal{F}$, and define $u := w(x, p)$. It suffices to show that p is repelled by x_u^y, because $\gamma \cdot p$ is attracted to x_v^y is equivalent to $\gamma^{-1} \cdot p$ is repelled by x_v^y. Define $L = l(u)$. We prove the theorem using induction on L.

First, we consider our base cases where $L = 0$ and $L = 1$. If $L = 0$, then $u = e$ so $p = x_e^y = x$, which is a fixed point of γ by Prop. 3.15. So p is repelled by $x_u^y = x$.

Next, suppose $L = 1$. Then $u = s_k$ for some $1 \leq k \leq n$. So by Prop. 3.16, then p is repelled by $x_{s_k}^y$.

Suppose for some $L_0 \geq 1$ that $L \leq L_0$ implies that p is repelled by x_u^y. Inductive step: Suppose $L = L_0 + 1$. Then $L \geq 2$, so u is an upper cover of at least two elements $v, w \in W$, and at least one is of the form $v = s_j u$.

Case 1: $w = s_k u$ for some $k \neq j$. Then x_u^y is j-adjacent to x_v^y and k-adjacent to x_w^y. Recall that

$$\alpha_u(x) = \bigcup_{f \in \alpha_v(x)} \alpha_j(f) \tag{92}$$

$$= \bigcup_{f \in \alpha_w(x)} \alpha_k(f). \tag{93}$$

Then there exist unique $p_v \in \alpha_v(x)$ and $p_w \in \alpha_w(x)$ such that p is j-adjacent to p_v and k-adjacent to p_w. In other words, $p \in \alpha_j(p_v) \cap \alpha_k(p_w)$, and since $j \neq k$ then the intersection of these distinct rational curves is a single point by Prop. 3.13. Clearly $w(x, p_v) = v$ because otherwise $w(x, p) = s_j w(x, p_v) \neq s_j w = u$. Similarly $w(x, p_w) = w$. Then $l(v) = l(w) = L_0$,

so by our inductive assumption then p_v is repelled by x_v^y and p_w is repelled by x_w^y. Then

$$\lim_{i\to\infty}\{\rho(\gamma^{-i})\cdot p\} = \lim_{i\to\infty}\alpha_j(\rho(\gamma^{-i})\cdot p_v)\cap\alpha_k(\rho(\gamma^{-i})\cdot p_w) \tag{94}$$

$$= \alpha_j(x_v^y)\cap\alpha_k(x_w^y) \tag{95}$$

$$= \{x_u^y\}. \tag{96}$$

Case 2: $w \neq s_k u$ for any $1 \leq k \leq n$. Then x_u^y and x_v^y are j-adjacent, but x_u^y and x_w^y are not adjacent, so the Richardson 1-variety containing them is a different type of span identified by the relative position $w(x_u^y, x_w^y)$.

Define $p_u \in \alpha_j(x_u^y)$ and $p_w \in \alpha_w(x)$ as the unique elements such that $p \in \mathrm{span}(p_u, p_w)$ where the span is of the same type as the Richardson 1-variety $\mathrm{span}(x_u^y, x_w^y) = \alpha_u(x)\cap\alpha_{\iota(w)}(y)$, i.e. the relative position $w(p_u, p_w) = w(x_u^y, x_w^y)$. Define p_v as in Case 1. Then $w(x, p_u) = u$ and $w(x, p_w) = w$, because otherwise their relative position would differ from that of x_u^y and x_w^y. Then p_u is repelled by x_u^y because it is contained in a j-span Richardson 1-variety, and p_w is repelled by x_w^y by our inductive assumption.

Then $p \in \alpha_j(p_v)\cap\mathrm{span}(p_u, p_w)$ by construction, and the intersection is a single point by Prop. 3.13 since the rational curves are of different types. Then

$$\lim_{i\to\infty}\{\rho(\gamma^{-i})\cdot p\} = \lim_{i\to\infty}\alpha_j(\rho(\gamma^{-i})\cdot p_v)\cap\mathrm{span}(\rho(\gamma^{-i})\cdot_u, \rho(\gamma^{-i})\cdot p_w) \tag{97}$$

$$= \alpha_j(x_v^y)\cap\mathrm{span}(x_u^y, x_w^y) \tag{98}$$

$$= \{x_u^y\}. \tag{99}$$

So p is repelled by x_u^y, and we are done. □

CHAPTER 4

Hyperconvexity in semisimple G

The goal of this chapter is to introduce a notion of hyperconvexity and partial hyperconvexity in general semisimple Lie groups using Richardson varieties. We show that these notions align with the existing notions of hyperconvexity and partial hyperconvexity in the setting of $SL_{n+1}\mathbb{C}$.

1. Hyperconvexity in semisimple G

Let G be a complex semisimple Lie group of rank $n \geq 2$. Let $\mathcal{F} = G/B$ be the associated flag variety, where $B \subset G$ is a minimal parabolic (Borel) subgroup.

Let Γ be a finitely generated word hyperbolic group with boundary $\partial\Gamma$. Recall that a Borel Anosov representation $\rho : \Gamma \to G$ admits a continuous, transverse, ρ-equivariant, dynamics preserving boundary map $\xi : \partial\Gamma \to \mathcal{F}$. We call the image of ξ the *limit set* $\Lambda \subset \mathcal{F}$.

Definition 4.1. Let $\rho : \Gamma \to G$ be a Borel Anosov representation. Suppose for all distinct $x, y, z \in \Lambda$ we have

$$\mathrm{Rich}^0_{x,y} \cap \mathrm{Rich}^0_{x,z} = \{x\}. \tag{100}$$

Then we say ρ is *hyperconvex*.

Using the Bruhat order structure of Richardson 0-varieties, we simplify the conditions in Def. 4.1 as follows:

Proposition 4.2. *Let $x, y, z \in \mathcal{F}$ be pairwise transverse. Then $Rich^0_{x,y} \cap Rich^0_{x,z} = \{x\}$ if and only if $x^y_{s_k} \neq x^z_{s_k}$ for all $1 \leq k \leq n$.*

The proof is provided in Section 4.

2. Partial hyperconvexity in semisimple G

We introduce two notions of partial hyperconvexity, and prove that they are equivalent in a large class of cases.

Consider $G = SL_{n+1}\mathbb{C}$. Then we can identify the simple roots with $\{1, \ldots, n\} = \Delta$. Let S be a symmetric subset of Δ, i.e. $S \subset \Delta$ and $S = S^*$.

DEFINITION 4.3. Let $1 \leq k \leq n$ and let $\rho : \Gamma \to SL_{n+1}\mathbb{C}$ be a $\{k-1, k, k+1\}$-Anosov representation with limit set $\Lambda \subset \mathcal{F}_{k-1,k,k+1}$. Suppose that for all $x, y, z \in \Lambda$ we have

$$(x_{k-1} + y_{n+1-k} \cap x_{k+1}) \cap (x_{k-1} + z_{n+1-k} \cap x_{k+1}) = x_{k-1}. \tag{101}$$

Then ρ is *k-hyperconvex*.

For each $s_k \in W$, define $S_k \subset \Delta$ as follows:

$$S_k := \{s_j \in \Delta : |s_j s_k| \neq 2\}. \tag{102}$$

In other words, S_k consists of s_k and the generating reflections which do not commute with s_k. In the Dynkin diagram, these are the vertices connected to s_k. Again, we assume that S_k is symmetric.

DEFINITION 4.4. Let $1 \leq k \leq n$ and let $\rho : \Gamma \to G$ be an S_k-Anosov representation with limit set $\Lambda \subset \mathcal{F}_k$. Suppose for all $x, y, z \in \Lambda$ we have

$$x^y_{s_k} \neq x^z_{s_k}. \tag{103}$$

Then ρ is *$\{s_k\}$-hyperconvex*.

We show that these definitions are equivalent in the setting where they both apply.

PROPOSITION 4.5. *If $\rho : \Gamma \to SL_{n+1}\mathbb{C}$ then ρ is $\{s_k\}$-Anosov if and only if ρ is k-Anosov.*

The proof is provided in Section 5.

In Section 7, we provide an explicit incidence geometry description of $\{s_k\}$-hyperconvexity for simple G of types A, B, C, D, and G_2 in Section 7. With the exception of two roots in type D, we see that $\{s_k\}$-hyperconvexity imples k-hyperconvexity in an appropriate $G < SL_N\mathbb{C}$. We also describe the semisimple case of type $A_1 \oplus A_1$.

REMARK 4.6. *If ρ is $\{s_k\}$-hyperconvex for all $1 \leq k \leq n$, then ρ is hyperconvex as in Def. 4.1.*

We also highlight the special case where G is semisimple.

PROPOSITION 4.7. *If G is semisimple with simple components $G_1, \dots, G_N$ then $\rho : \Gamma \to G$ is hyperconvex if and only if $\rho_i : \Gamma \to G_i$ is hyperconvex for all $1 \leq i \leq N$.*

PROOF. Observe that the Dynkin diagram of G has N connected components, and that each connected component is associated with some simple G_i. Consider $\rho_i : \Gamma \to G_i$, and let $s_k \in \Delta_i$ and s_{i_k} refer to the corresponding generator in Δ. Then $S_k = S_{i_k} \cap \Delta_i$, so ρ is S_{i_k} Anosov and $\{s_{i_k}\}$-hyperconvex if and only if ρ_i is S_k Anosov and $\{s_k\}$-hyperconvex. □

In particular, if G is semisimple and not simple, even if the simple components are all of type A, $\{s_k\}$-hyperconvexity does not generally imply k-hyperconvex in some $G < SL_N\mathbb{C}$.

3. Properties of hyperconvex representations

We show that our notion of hyperconvex representations have desireable properties, in particular,

PROPOSITION 4.8. *The set of $\{s_k\}$-hyperconvex representations is open in $Hom(\Gamma, G)$.*

Furthermore, let $x \in \Lambda$ and $1 \leq k \leq n$, and consider the limit set projection

$$\pi_k^x : \Lambda \to \alpha_k(x) \cong \mathbb{CP}^1 \tag{104}$$

given by

$$\pi_k^x(y) = x_{s_k}^y : y \neq x \tag{105}$$

$$\pi_k^x(x) = x. \tag{106}$$

PROPOSITION 4.9. *If ρ is $\{s_k\}$-hyperconvex, then for all $x \in \Lambda$ the projection map $\pi_k^x : \Lambda \to \alpha_k(x)$ is injective and continuous on Λ.*

These properties are crucial to recent research on hyperconvex representations in $SL_{n+1}\mathbb{C}$, e.g. in [**18, 19, 32**], and their proofs are provided in Section 6. For G of type A_n, these properties were proved in [**32**].

As a direct consequence of Proposition 4.9, using that the only continuous and injective maps from $\mathbb{CP}^1$ to $\mathbb{CP}^1$ are surjective, we obtain

COROLLARY 4.10. *If $\partial\Gamma \cong \mathbb{CP}^1$ then $\pi_k^x : \Lambda \to \alpha_k(x)$ is surjective for all $x \in \Lambda$.*

To prove Proposition 4.9, we use the following lemma about hyperbolic groups, which is likely already known but will be useful for us.

LEMMA 4.11. *Let $\gamma \in \Gamma$ be an element of infinite order, with fixed points $\gamma^+, \gamma^- \in \partial\Gamma$. Then $\langle\gamma\rangle$ acts cocompactly on $\partial\Gamma \setminus \{\gamma^+, \gamma^-\}$.*

The proof is also provided in Section 6.

4. Richardson intersections

In this section we prove Proposition 4.2. Let $x, y, z \in \mathcal{F}$ be pairwise transverse. Then $R_x^{y,z} = \mathrm{Rich}_{x,y}^0 \cap \mathrm{Rich}_{x,z}^0$.

PROPOSITION 4.12. *If $p \in R_x^{y,z}$ then $p = x_w^y = x_w^z$ for some $w \in W$. That is, the labelling of p as as an element of W relative to (x, y) and (x, z) is the same.*

PROOF. Let $p \in R_x^{y,z}$. Then $p \in \mathrm{Rich}_{x,y}^0$ so $p = x_w^y$ for some $w \in W$. Similarly, $p \in \mathrm{Rich}_{x,z}^0$ so $p = x_u^z$ for some $u \in W$. Then the relative position of x and p is w, since $p \in \alpha_w(y)$ but not in any smaller principle Schubert variety, and by a similar argument the relative position of x and p is u. So $u = w$ and $p = x_w^y = x_w^z$. □

So we may describe $R_x^{y,z}$ using a subset of W. Define

$$I_x^{y,z} = \{w \in W : x_w^y = x_w^z\}. \tag{107}$$

PROPOSITION 4.13. *Then $I_x^{y,z}$ is a left Bruhat ideal.*

PROOF. It suffices to show that $e \in I_x^{y,z}$, and that if $u \in I_x^{y,z}$ is a left-upper cover of $v \in W$ then $v \in I_x^{y,z}$. This is because $u >_L v$ if there exists a sequence $v < s_{k_1}v < \cdots < s_{k_N} \cdots s_{k_1}v = u$.

Clearly $x = x_e^y = x_e^z \in R_x^{y,z}$ so $e \in I_x^{y,z}$. Let $u \in I_x^{y,z}$ and let $u >_L v$ such that $u = s_k v$. Then $x_u^y = x_u^z$ and x_u^y, x_v^y, x_v^z are k-adjacent. By construction $x_u^y, x_u^z \in \alpha_u(x)$. Additionally, $\iota(s_k v) = s_k \iota(v) = \iota(u)$. So $\alpha_{\iota(v)}(y)$ contains any points which are k-adjacent to points in $\alpha_{\iota(u)}(y)$, and therefore $x_v^y, x_v^z \in \alpha_{\iota(v)}(y)$. But the intersection $\alpha_v(x) \cap \alpha_{\iota(v)}(y)$ is a single point, so $x_v^y = x_v^z$, and $v \in I_x^{y,z}$. □

Since $I_x^{y,z}$ is a left Bruhat ideal, then $s_1, \ldots, s_n \notin I_x^{y,z}$ means that $I_x^{y,z} = \{e\}$ and therefore $\mathrm{Rich}_{x,y}^0 \cap \mathrm{Rich}_{x,z}^0 = \{x\}$. This completes the proof of Prop. 4.2.

5. Proof of Proposition 4.5

In this section we prove that if G is type A_n, then s_k-hyperconvexity is equivalent to k-hyperconvexity.

Proof of Proposition 4.5. Let $1 \leq k \leq n$ and consider the flag $x_{s_k}^y$. Since this point is k-adjacent to x, then only the k-th subspace differs from that of x. Additionally, $x_{s_k}^y$ must not be transverse to y, i.e. the k-th dimensional subspace must intersect y_{n+1-k} nontrivially. Then

$$x_{s_k}^y = (x_1, \ldots, x_{k-1}, \widetilde{x_k}, x_{k+1}, \ldots, x_n), \tag{108}$$

where

$$\widetilde{x_k} \cap y_{n+1-k} \neq \{0\}. \tag{109}$$

Observe that x_{k+1} and y_{n+1-k} have a 1-dimensional intersection, and $x_{k-1} \cap y_{n+1-k} = \{0\}$. So then

$$\widetilde{x_k} = x_{k-1} + x_{k+1} \cap y_{n+1-k}. \tag{110}$$

We also consider the corresponding definition for $x_{s_k}^z$. If $s_k \in I_x^{y,z}$, then $x_{s_k}^y = x_{s_k}^z$ and

$$x_{k-1} + x_{k+1} \cap y_{n+1-k} = x_{k-1} + x_{k+1} \cap z_{n+1-k}, \tag{111}$$

and if $s_k \notin I_x^{y,z}$, then

$$(x_{k-1} + x_{k+1} \cap y_{n+1-k}) \cap (x_{k-1} + x_{k+1} \cap z_{n+1-k}) = x_{k-1}. \tag{112}$$

We have shown that $\{s_k\}$-hyperconvexity and k-hyperconvexity are equivalent in $SL_{n+1}\mathbb{C}$. □

6. Proof of Propositions 4.8 and 4.9

First, we prove Lemma 4.11.

Proof of Lemma 4.11. We construct a quasi-fundamental domain, i.e. a compact subset whose orbit under $\langle\gamma\rangle$ covers $\partial\Gamma \setminus \{\gamma^+, \gamma^-\}$ but self-intersects finitely many times. This suffices because $\partial\Gamma$ is compact by Theorem 2.5.

Let U_+ and U_- be disjoint neighborhoods of γ^+ and γ^-, respectively, in $\partial\Gamma$. Then by Theorem 2.8 there exists some finite $k \geq 1$ such that $\gamma^k \cdot (\partial\Gamma \setminus U_-) \subset U_+$. Define

$$K = \partial\Gamma \setminus \{\gamma \cdot U_+ \cup \gamma^{-1} \cdot U_-\}. \tag{113}$$

Clearly K is compact, since it is a closed subset of a compact set.

Let $p \in \partial\Gamma \setminus \{\gamma^+, \gamma^-\}$. Claim: $\langle\gamma\rangle \cdot p$ intersects K a finite, nonzero number of times. Then either $p \in \gamma \cdot U_+$, $p \in K$, or $p \in \gamma^{-1}U_-$. First, we show finiteness: let $p \in K$. Then $\gamma^N \cdot p$ is in $\gamma \cdot U_+$ for any $N \geq k+1$ and in $\gamma^{-1} \cdot U_-$ for any $N < -k-1$, thus $\langle\gamma\rangle \cdot p \cap K$ is finite. Next, we show that the intersection is nonempty. Let $p \in \gamma^{-1} \cdot U_-$. By Theorem 2.8 there exists some minimum power $j \geq 1$ such that $\gamma^j \cdot p \in U_+$. Thus $\gamma^j \cdot p \in (U_+ \setminus \gamma \cdot U_+) \subset K$. A similar argument applies if $p \in \gamma \cdot U_+$, using a negative power of γ.

So then $\langle\gamma\rangle \cdot K = \partial\Gamma \setminus \{\gamma^+, \gamma^-\}$, and K is a quasi-fundamental domain. Thus the action of $\langle\gamma\rangle$ on $\Gamma \setminus \{\gamma^+, \gamma^-\}$ is cocompact. □

Next, we prove Proposition 4.9. Observe that the projection $\pi_k^x : \Lambda \to \alpha_k(x)$ is continuous on $\Lambda \setminus \{x\}$ by Proposition 3.7.

PROOF OF PROPOSITION 4.9. If Γ is elementary, e.g. Γ finite or $\partial\Gamma = \{x, y\}$, then the result is trivial.

Suppose x is $\xi(\gamma^+)$ for some infinite order element $\gamma \in \Gamma$. It suffices to show that the projection $\pi_k^x : \Lambda \to \alpha_k(x)$ given by $y \mapsto x_k^y$ and $x \mapsto x$ is continuous at x. Define $y := \xi(\gamma^-)$. Let $\{x(i)\}_{i=1}^{\infty} \subset \Lambda \setminus \{x, y\}$ such that $\lim_{i\to\infty} x(i) = x$. We show that

$$\lim_{i\to\infty} \pi_k^x(x(i)) = \lim_{i\to\infty} x_k^{x(i)} \tag{114}$$

$$= x. \tag{115}$$

Consider the action of $\rho(\gamma)$ on $\Lambda \setminus \{x, y\}$, which by Lemma 4.11 is cocompact. Let $U \subset \Lambda \setminus \{x, y\}$ be a fundamental domain which contains $x(1)$. Then $\overline{U}$ does not contain x or y. Then define

$$V(i) := \bigcup_{j=i}^{\infty} \rho(\gamma^j) \cdot \overline{U}. \tag{116}$$

Possibly by passing to a subsequence, we may assume that $x(i) \in V(i)$ for all i.

Identify $\alpha_k(x)\backslash\{x_{s_k}^y\} \cong \mathbb{CP}^1\backslash\{\infty\}$ with the complex plane $\mathbb{C}$ via a sterographic projection from $x_{s_k}^y$ which sends $x \mapsto 0$, equipped with the Euclidean metric $|\cdot|$. We observe that $\pi_k^x(V(i))$ is bounded, so define

$$d(i) := \sup_{z\in V(i)} |x_{s_k}^z|. \tag{117}$$

Then in particular, $|x_{s_k}^{x(i)}| \leq d(i)$.

By Prop. 3.16 then $\rho(\gamma)$ acts loxodromically on $\alpha_k(x)$ with attracting fixed point x. So there is a constant $R > 1$ such for all $z \in \alpha_k(x) \setminus \{x^y_{s_k}\}$ then $|z| = R|\rho(\gamma) \cdot z|$. Then

$$d(i) = \sup_{z \in V(i)} |x^z_{s_k}| \tag{118}$$

$$= \sup_{z \in V(i)} R|x^{\rho(\gamma)\cdot z}_{s_k}| \tag{119}$$

$$= R \sup_{z \in V(i+1)} |x^z_{s_k}| \tag{120}$$

$$= Rd(i+1). \tag{121}$$

So then

$$|x^{x(i)}_{s_k}| \leq \left(\frac{1}{R}\right)^{i-1} d(1). \tag{122}$$

Then $\lim_{i\to\infty} |x^{x(i)}_{s_k}| = 0$ and $\lim_{i\to\infty} x^{x(i)}_{s_k} = x$. So π^x_k is continuous at x.

We observe that any neighborhood U of x in $\mathcal{F}$, then since $\pi^x_k : \Lambda \cap U \to \alpha_k(x)$ is a continuous function with fixed point x, then there is a smaller neighborhood $V \subset U$ such that $\pi^x_k(V \cap \Lambda) \subset U \cap \alpha_k(x)$.

Next, suppose $y \in \Lambda$ is not a fixed point of an infinite order element. Let $\{y(i)\}_{i=1}^{\infty} \subset \Lambda \setminus \{y\}$ such that $\lim_{i\to\infty} y(i) = y$. Then $\pi^y_k(y(i)) \in \alpha_k(y)$ for all i, and $\alpha_k(y)$ is compact, so the sequence $\{y(i)\}_{i=1}^{\infty}$ has a limit point $z \in \alpha_k(xy)$. Clearly z is not in $\pi^y_k(\Lambda \setminus \{y\})$ because $y(i) \to y$ and π^y_k is continuous and injective on $\Lambda \setminus \{x\}$. Let U be a neighborhood of y in $\mathcal{F}$. Then there is a sub-neighborhood $V \subset U$ of y such that for any $x \in \Lambda \cap V$ rational (i.e. $x = \xi(\gamma^+)$ for some infinite-order $\gamma \in \Gamma$) we have $\pi^x_k(y) \in U$. Since $y(i) \to y$, $\pi^x_k(y(i))$ is eventually in U. The limit point z is in the closure of the limits $\pi^x_k(y(i)$, and since these are in U, then $z \in \overline{U}$. Since U was an arbitrary neighborhood of y, then $z = y$. Since $y^{y(i)}_k$ has a unique limit point, then the sequence converges to y, i.e.

$$\lim_{i\to\infty} \pi^y_k(y(i)) = \lim_{i\to\infty} y^{y(i)}_k \tag{123}$$

$$= y. \tag{124}$$

Since y was arbitrary, then $\pi_k^y : \Lambda \to \alpha_k(y)$ is continuous for all $y \in \Lambda$. □

Finally, we prove that $\mathrm{Hyp}(\Gamma, G) \subset \mathrm{Hom}(\Gamma, G)$ is open.

Proof of Proposition 4.8. Suppose $\rho \in \mathrm{Hom}(\Gamma, G)$ is $\{s_k\}$-hyperconvex. Then ρ is S_k-Anosov, and since Anosov is an open condition, there is an open neighborhood U of ρ_0 in $\mathrm{Hom}(\Gamma, G)$ which is also S_k-Anosov.

Observe that for any distinct triple $x, y, z \in \Lambda$, then $\pi_k^x(y) = \pi_k^x(z)$ if and only if

$$\pi_k^{\rho(\gamma)\cdot x}(\rho(\gamma)\cdot y) = \pi_k^{\rho(\gamma)\cdot x}(\rho(\gamma)\cdot z) \tag{125}$$

Since $\rho(\Gamma)$ is a uniform convergence group by Prop. 2.7, it acts properly discontinuously and cocompactly on distinct triples in Λ. Thus there are closed subsets $X, Y, Z \subset \Lambda$ with X, Y disjoint such that for any distinct $x, y, z \in \Lambda$ there is some $\gamma \in \Gamma$ such that $\rho(\gamma)\cdot x \in X$, $\rho(\gamma)\cdot y \in Y$, and $\rho(\gamma)\cdot z \in Z$. Since ρ is hyperconvex, then for any $x \in X$, $\pi_k^x(Y)$ and $\pi_k^x(Z)$ are disjoint in $\alpha_k(x)$.

Let $p \in \Lambda \setminus \{X \cup Y \cup Z\}$ and $y \in Y$. For each $x \in X$, we may identify $\alpha_k(x) \setminus \{x_{s_k}^p\}$ with the complex plane $\mathbb{C}$ via a sterographic projection from $x_{s_k}^y$ which sends $x \mapsto 0$, and $x_{s_k}^y \mapsto 1$, and let d_x be the Euclidean metric $|\cdot|$. Then $\{d_x\}_{x\in X}$ is a continuous family of metrics on $\alpha_k(x)$, and

$$D_X(Y, Z) := \inf_{x\in X} d_x(\pi_k^x(Y), \pi_k^x(Z)) \tag{126}$$

is positive.

The boundary map ξ_t varies continuously with respect to $\rho_t \in U$, as does the limit set $\Lambda_t \subset \mathcal{F}_k$. Define $X_t = \xi_t \circ \xi^{-1}(X)$, $Y_t = \xi_t \circ \xi^{-1}(Y)$, and $Z_t = \xi_t \circ \xi^{-1}(Z)$. As before, we may assume that any distinct triple in Λ_t is conjugate to some triple $x_t, y_t, z_t \in \Lambda_t$ where $x_t \in X_t$, $y_t \in Y_t$, and $z_t \in Z_t$. For a sufficiently small perturbation ρ_t of ρ in U, e.g. one in which $d_{x_t}(y_t, z_t)$ changes by less than half of $D_X(Y, Z)$, then $\pi_k^{x_t}(Y_t)$ and $\pi_k^{x_t}(Z_t)$ are disjoint for all $x_t \in X_t$. Thus $\pi_k^{x_t}(y_t) \neq \pi_k^{x_t}(z_t)$ for any distinct $x_t, y_t, z_t \in \Lambda_t$ and ρ_t is k-hyperconvex.

Thus the set of $\{s_k\}$-hyperconvex representations is open in $\mathrm{Hom}(\Gamma, G)$. □

7. Incidence geometry in types A, B, C, D, and G_2

The remainder of this chapter is devoted to computing $x^y_{s_k}$ explicitly for simple G of types A, B, C, D, and G_2. We also include the semisimple type $A_1 \oplus A_1$.

7.1. Type A_n. Let G be of type A_n for $n \geq 1$, e.g. $G = SL_{n+1}\mathbb{C}$. Then flags consist of n-tuples $x = (x_1, \ldots, x_n)$ where each x_k is a k-dimensional linear subspace of $\mathbb{C}^{n+1}$, and $x_k \subset x_{k+1}$.

Two flags $x, y \in \mathcal{F}$ are transverse if for all $1 \leq k \leq n$, the sum $x_k + y_{n+1-i}$ is direct.

Then the Richardson 0-variety $x^y_{s_k}$ is given by

$$x^y_{s_k} = (x_1, \ldots, x_{k-1}, \widetilde{x_k}, x_{k+1}, \ldots, x_n), \tag{127}$$

where

$$\widetilde{x_k} = x_{k-1} + x_{k+1} \cap y_{n+1-k}. \tag{128}$$

When $k = 1$, we use $x_0 = \{0\}$, and when $k = n$ we use $x_{n+1} = \mathbb{C}^{n+1}$.

7.2. Type B_n. Let G be of type B_n for $n \geq 2$, e.g. $G = SO_{2n+1}\mathbb{C}$. Then full flags consist of n-tuples $(x_1, \ldots, x_n)$ where each x_k is a k-dimensional isotropic subspace of $\mathbb{C}^{2n+1}$, and $x_k \subset x_{k+1}$. The nondegenerate symmetric bilinear form b that is used to define $SO_{2n+1}\mathbb{C}$ induces a perpendicular operator:

$$x_k^\perp := \{v \in \mathbb{C}^{2n+1} : b(v, w) = 0 \text{ for all } w \in x_k\}. \tag{129}$$

Then $x_k^\perp$ has dimension $2n + 1 - k$ and contains x_k.

Two flags x, y are transverse if for all $1 \leq k \leq n$ the sum $x_k + y_k^\perp$ is direct. Then the 0-Richardson variety $x^y_{s_k}$ is given by

$$x^y_{s_k} = (x_1, \ldots, x_{k-1}, \widetilde{x_k}, x_{k+1}, \ldots, x_n), \tag{130}$$

where for $1 \leq k \leq n-1$ we have

$$\widetilde{x_k} = x_{k-1} + y_k^\perp \cap x_{k+1}, \tag{131}$$

and for $k = n$ we have

$$\widetilde{x_n} = x_{n-1} + y_k^\perp \cap x_n^\perp. \tag{132}$$

So any flag $x = (x_1, \ldots, x_n) \in \mathcal{F}(B_n)$ may be uniquely extended to a full flag $(x_1, \ldots, x_n, x_n^\perp, \ldots, x_1^\perp) \in \mathcal{F}(A_{2n})$. Similarly, ρ which are S_k-Anosov in B_n are $\{k-1, k, k+1, 2n+2-k, 2n+1-k, 2n-k\}$-Anosov in A_{2n}, and $\{s_k\}$-hyperconvex in B_n implies $\{k, 2n+1-k\}$-hyperconvex in A_{2n}.

7.3. Type C_n. Let G be of type C_n for $n \geq 2$, e.g. $G = Sp_{2n}\mathbb{C}$. Then flags consist of n-tuples $(x_1, \ldots, x_n)$ where each x_k is a k-dimensional isotropic subspace of $\mathbb{C}^{2n}$ and $x_k \subset x_{k+1}$. The nondegenerate, skew-symmetric bilinear form c used to define $Sp_{2n}\mathbb{C}$ induces a perpendicular operator:

$$x_k^\perp := \{v \in \mathbb{C}^{2n} : c(v, w) = 0 \text{ for all } w \in x_k\}. \tag{133}$$

Then $x_k^\perp$ has dimension $2n-k$ and contains x_k, and $x_n = x_n^\perp$.

Two flags x, y are transverse if for all $1 \leq k \leq n$ the sum $x_k + y_k^\perp$ is direct. Then the Richardson 0-variety $x_{s_k}^y$ is given by

$$x_{s_k}^y = (x_1, \ldots, x_{k-1}, \widetilde{x_k}, x_{k+1}, \ldots, x_n), \tag{134}$$

where for $1 \leq k \leq n-1$ we have

$$\widetilde{x_k} = x_{k-1} + y_k^\perp \cap x_{k+1}, \tag{135}$$

and for $k = n$ we have

$$\widetilde{x_n} = x_{n-1} + y_k^\perp \cap x_{n-1}^\perp. \tag{136}$$

In particular, each full flag $x = (x_1, \ldots, x_n) \in \mathcal{F}(C_n)$ may be uniquely extended to a full flag $(x_1, \ldots, x_n = x_n^\perp, \ldots, x_1^\perp) \in \mathcal{F}(A_{2n-1})$. Similarly, ρ which are S_k-Anosov in C_n are also $\{k-1, k, k+1, 2n-2-k, 2n-1-k, 2n-k\}$-Anosov in A_{2n-1}. So $\{s_k\}$-hyperconvex in C_n implies $\{k, 2n-1-k\}$-hyperconvex in A_{2n-1}.

7.4. Type D_n. Let G be of type D_n for $n \geq 4$, e.g. $G = SO_{2n}\mathbb{C}$. Then flags consist of n-tuples $(x_1, \ldots, x_n)$ where for $1 \leq k \leq n-2$ then x_k is a k-dimensional linear subspace of $\mathbb{C}^{2n}$, and x_{n-1}, x_n are n dimensional subspaces. For $1 \leq k \leq n-3$ then incidence is familiar: $x_k \subset x_{k+1}$. However, it differs for the remaining subspaces: $x_{n-1} \cap x_n$ has dimension $n-1$, and $x_{n-2} \subset (x_{n-1} \cap x_n)$. The intersection $x_{n-1} \cap x_n$ uniquely determines x_{n-1} and x_n.

The nondegenerate symmetric bilinear form d used to define $SO_{2n}\mathbb{C}$ induces a perpendicular operator:

$$x_k^\perp := \{v \in \mathbb{C}2n : d(v, w) = 0 \text{ for all } w \in x_k\}. \tag{137}$$

Then $x_k^\perp$ contains x_k and has dimension $2n - \dim(x_k)$. Furthermore, $(x_{n-1} \cap x_n)^\perp = x_{n-1} + x_n$ and has dimension $n+1$.

Two flags are transverse if for all $1 \leq k \leq n$ the sum $x_k + y_k^\perp$ is direct. Then the Richardson 0-variety $x_{s_k}^y$ is given by

$$x_{s_k}^y = (x_1, \ldots, x_{k-1}, \widetilde{x_k}, x_{k+1}, \ldots, x_n), \tag{138}$$

where for $1 \leq k \leq n-3$ we have

$$\widetilde{x_k} = x_{k-1} + y_k^\perp \cap x_{k+1} \text{ for } 1 \leq k \leq n-3. \tag{139}$$

However, for $k = n-2$, we use the containment of $x_{n-2} \subset x_{n-1} \cap x_n$, so

$$\widetilde{x_{n-2}} = x_{n-3} + y_{n-2}^\perp \cap (x_{n-1} \cap x_n). \tag{140}$$

Finally, for $k = n-1$ or $k = n$, we have

$$\widetilde{x_{n-1}} = x_{n-2} + y_{n-1} \cap x_{n-2}^{\perp} \tag{141}$$

$$\widetilde{x_n} = x_{n-2} + y_n \cap x_{n-2}^{\perp}. \tag{142}$$

Since x and y are transverse, the subspace $y_n \cap x_{n-2}^{\perp}$ is 2-dimensional, so the definition P_n-Anosov in D_n implies $(n-2, n, n+2)$-Anosov in A_{2n-1} rather than $(n-1, n, n+1)$-Anosov in A_{2n-1}. So $\{s_n\}$-hyperconvex in D_n does not imply n-hyperconvex in A_{2n-1}. Similarly, s_{n-1}-hyperconvex in D_n does not imply n- or $n-1$-hyperconvexity in A_{2n-1}.

By construction each $x_{s_k}^y$ is k-adjacent to x and not transverse to y. Flags $x = (x_1, \ldots, x_n) \in \mathcal{F}(D_n)$ may be extended to a full flag in $\mathcal{F}(A_{2n-1})$ in two ways:

$$(x_1, \ldots, x_{n-2}, x_{n-1} \cap x_n, x_{n-1}, x_{n-1} + x_n, x_{n-2}^{\perp}, \ldots, x_1^{\perp}) \tag{143}$$

$$(x_1, \ldots, x_{n-2}, x_{n-1} \cap x_n, x_n, x_{n-1} + x_n, x_{n-2}^{\perp}, \ldots, x_1^{\perp}). \tag{144}$$

So in particular, ρ $\{s_{n-1}, s_n\}$-hyperconvex in D_n implies that ρ is $\{n-2, n-1, n, n+1\}$-Anosov in A_{2n-1} (in two ways) and $\{n-1, n\}$-hyperconvex (in two ways).

7.5. Type G_2. Let G be of type G_2. Then full flags consist of 2-tuples $x = (x_1, x_2)$ where each x_k is a k-dimensional subspace of $\mathbb{C}^7$. There is a degenerate, symmetric bilinear form g on $\mathbb{C}^7$ preserved by G. The incidence relationship is as follows: $x_1 \subset x_2 \subset (x_1)^a$ where $(x_1)^a$ is the annihilator i.e.

$$(x_1)^a = \{v \in \mathbb{C}^7 : g(v, x_1) = 0\}. \tag{145}$$

Thus $g(x_1, x_1) = 0$ and $g(x_1, v) = 0$ for any $v \in x_2$ as well. This is explained more thoroughly in [**13**].

Since $\mathfrak{g}_2$ is also subalgebra of $\mathfrak{so}_7\mathbb{C}$, then as in type B_n, there is an orthogonal operator $\perp$ taking d-dimensional subspaces to $(7-d)$-dimensional subspaces.

Two flags x, y are transverse if the sums $x_1 + y_1^\perp$ and $x_2 + x_2^\perp$ are direct. Then the Richardson 0-varieties $x^y_{s_1}$ and $x^y_{s_2}$ are given by

$$x^y_{s_1} = (y_1^\perp \cap x_2, x_2) \tag{146}$$

$$x^y_{s_2} = (x_1, y_2^\perp \cap (x_1)^a). \tag{147}$$

Then a flag $x = (x_1, x_2) \in \mathcal{F}(G_2)$ uniquely extend to a flag $(x_1, x_2, x_1^a) \in \mathcal{F}(B_3)$ and therefore uniquely extend to a flag

$$(x_1, x_2, x_1^a, (x_1^a)^\perp, x_2^\perp, x_1^\perp) \in \mathcal{F}(A_6). \tag{148}$$

Observe that $S_1 = S_2 = \Delta$, and Borel Anosov representations in G_2 are Borel Anosov in $SO_7\mathbb{C}$ and $SL_7\mathbb{C}$, Thus $\{s_1\}$- and $\{s_2\}$-hyperconvex in G_2 are equivalent to $\{s_1, s_3\}$- and $\{s_2\}$-hyperconvex in B_3 and therefore $\{1, 3, 4, 6\}$- and $\{2, 5\}$-hyperconvex in A_6, respectively.

7.6. Type $A_1 \oplus A_1$. Finally, we consider a semisimple case, to better illustrate Proposition 4.7.

Let G be of type $A_1 \oplus A_1$, e.g. $G = SL_2\mathbb{C} \times SL_2\mathbb{C}$. Then full flags consist of 2-tuples $x = (x_1, x_2)$, where each x_k is an element of (a separate copy of) $\mathbb{CP}^1$. There is no additional incidence requirement between x_1 and x_2.

Two flags $x, y \in \mathcal{F}$ are transverse if $x_1 \neq y_1$ and $x_2 \neq y_2$. Then

$$x^y_{s_1} = y^x_{s_2} = (y_1, x_2) \tag{149}$$

$$x^y_{s_2} = y^x_{s_1} = (x_1, y_2). \tag{150}$$

In other words, ρ is $\{s_1\}$-hyperconvex if the restriction to the first A_1 is Anosov, and ρ is $\{s_2\}$-hyperconvex if the restriction to the second A_1 is Anosov. Such representations, embedded into A_3 via block diagonals, would not be considered k-hyperconvex for any k.

CHAPTER 5

Hyperconvexity is preserved under topological conjugacy

The goal of this chapter is to describe properties of topological conjugacies between Anosov representations. In particular, we show that hyperconvexity is preserved by topological conjugacies and by certain partial topological conjugacies.

1. Notation

Let G be a complex semisimple Lie group of rank $n \geq 2$. Let $\mathcal{F} = G/B$ be the associated flag variety, where $B \subset G$ is a minimal parabolic (Borel) subgroup.

Let W be the Weyl group of G. Then W is equipped with a Bruhat order, and has a unique longest element $w_0 \in W$. Recall that *balanced ideals* are Bruhat order ideals $I \subset W$ such that $I^\perp = I$, where $I^\perp := w_0(W \setminus I)$.

Let Γ be a finitely generated word hyperbolic group with Gromov boundary $\partial\Gamma$. Recall that a *(Borel)-Anosov representation* is a homomorphism $\rho : \Gamma \to G$ such that there exists a continuous, transverse, ρ-equivariant, dynamics preserving boundary map $\xi : \partial\Gamma \to \mathcal{F}$. For ρ Anosov, the boundary map ξ is unique.

Let $\rho, \rho' : \Gamma \to G$ be Anosov representations. We use the following notation related to ρ and ρ'.

- $\xi, \xi' : \partial\Gamma \to \mathcal{F}$ are the boundary maps of ρ and ρ'.
- Λ, Λ' are the limit sets, i.e. the images of ξ and ξ' in $\mathcal{F}$.
- $Th^I(x) = \bigcup_{w \in I} \alpha_w(x)$ is the I-thickening of $x \in \mathcal{F}$ for some balanced ideal $I \subset W$.
- $Th^I(\Lambda) = \bigcup_{x \in \Lambda} Th^I(x)$ is the thickening of the limit curve.

We call ρ and ρ' *topologically conjugate* if there exists a homeomorphism $F : \mathcal{F} \to \mathcal{F}$ such that for all $\gamma \in \Gamma$ and all $x \in \mathcal{F}$, then $F(\rho(\gamma) \cdot x) = \rho'(g) \cdot F(x)$. We call F a *topological conjugacy* between ρ and ρ'.

Recall that an Anosov representations ρ is *hyperconvex* if for all distinct $x, y, z \in \Lambda$ then $\mathrm{Rich}^0_{x,y} \cap \mathrm{Rich}^0_{x,z} = \{x\}$.

2. Hyperconvexity is preserved under topological conjugacy

THEOREM 5.1. *If ρ and ρ' are topologically conjugate, then ρ is hyperconvex if and only if ρ' is hyperconvex.*

We use a supporting lemma to prove this theorem.

Suppose $F : \mathcal{F} \to \mathcal{F}$ is a topological conjugacy between ρ and ρ'. Then considering the action of Γ on $\partial\Gamma$, and that ξ and ξ' preserve dynamics, then for all $x \in \Lambda$ we have $F(x) = \xi' \circ \xi^{-1}x \in \Lambda'$. So $F(\Lambda) = \Lambda'$, and for any $x \in \Lambda$, we may use the notation $x' := F(x) \in \Lambda'$.

We say *F preserves the Richardson varieties of Λ* if for all distinct $x, y \in \Lambda$ and all $0 \leq k \leq l(w_0)$, then (as sets) $F(\mathrm{Rich}^k_{x,y}) = \mathrm{Rich}^k_{x',y'}$. In particular, a topological conjugacy which preserves Richardson varieties of Λ may send x^y_w to $x'^{y'}_u$ where $w \neq u \in W$.

Furthermore, if there exists a Bruhat order automorphism $\phi \in Aut(W)$ such that $F(x^y_w) = x'^{y'}_{\phi(w)}$ for all $w \in W$, then we say *F ϕ-matches the Richardson varieties of Λ*.

LEMMA 5.2. *If F is a topological conjugacy between ρ and ρ', then F preserves the Richardson varieties of Λ. Furthermore, there exists a Bruhat order automorphism $\phi \in Aut(W)$ such that F ϕ-matches the Richardson varieties of Λ.*

We highlight the following consequence of Lemma 5.2, of interest to those studying domains of discontinuity of Anosov representations.

COROLLARY 5.3. *Let $x \in \Lambda$ and $I \subset W$ a balanced ideal. Then $F(Th^I(x)) = Th^{\phi(I)}(x')$.*

This is because Schubert varieties are Richardson varieties of the form $\alpha_w(x) \cap \alpha_{w_0}(y)$, so if F ϕ-matches the Richardson varieties of Λ then F ϕ-matches any Bruhat ideal thickening of Λ.

EXAMPLE 5.4. *For the following balanced ideals, we have $F(Th^I(x)) = Th^I(x')$.*

- *For $W = A_2$ and I the unique balanced ideal $\{e, s_1, s_2\}$.*
- *Any W where $l(w_0)$ is odd (e.g. A_{2n}) and $I_{<\frac{1}{2}} = \{g \in W : l(g) < l(w_0)/2\}$.*
- *Any W such that $Aut(W)$ is trivial and any balanced $I \subset W$.*

3. Proof of Theorem 5.1

We first prove our main theorem assuming Lemma 5.2.

Proof of Theorem 5.1. Suppose $F : \mathcal{F} \to \mathcal{F}$ is a topological conjugacy between Anosov representations $\rho, \rho' : \Gamma \to G$.

Assume that ρ is hyperconvex. Let $x, y, z \in \Lambda$ distinct. By Def. 4.1 then $\mathrm{Rich}^0_{x,y} \cap \mathrm{Rich}^0_{x,z} = \{x\}$. By Lemma 5.2, then F preserves Richardson 0-varieties, and

$$F(\mathrm{Rich}^0_{x,y}) = \mathrm{Rich}^0_{x',y'}, \text{ and} \tag{151}$$

$$F(\mathrm{Rich}^0_{x,z}) = \mathrm{Rich}^0_{x',z'}. \tag{152}$$

Since F is a homeomorphism, then

$$F(\{x\}) = F(\mathrm{Rich}^0_{x,y} \cap \mathrm{Rich}^0_{x,z}) \tag{153}$$

$$= F(\mathrm{Rich}^0_{x,y}) \cap F(\mathrm{Rich}^0_{x,z}) \tag{154}$$

$$= \mathrm{Rich}^0_{x',y'} \cap \mathrm{Rich}^0_{x',z'} \tag{155}$$

$$= \{x'\}. \tag{156}$$

So in particular, x', y', z' are linearly independent. Since x, y, z were arbitrary, then this holds for all triples in $x', y', z' \in \Lambda'$, and ρ' is hyperconvex.

The other direction proceeds similarly. Use F^{-1} as a topological conjugacy between ρ' and ρ. Thus ρ' hyperconvex implies that ρ is hyperconvex. □

4. Hyperconvexity is preserved under I_Δ-topological conjugacy

Theorem 5.1 shows that topological conjugacy is an equivalence relation that preserves hyperconvexity. However, it is not clear that nontrivial examples of topological conjugacies exist in higher rank.

We introduce a less restrictive equivalence relation called a partial topological conjugacy. We show that the relation preserves hyperconvexity, and construct nontrivial examples. Let

$I \subset W$ be a Bruhat order ideal. The I-thickening $Th^I(\Lambda)$ is invariant and closed under the action of ρ.

DEFINITION 5.5. An *I-topological conjugacy* between $\rho, \rho' : \Gamma \to G$ is a topological conjugacy

$$F : Th^I(\Lambda) \to Th^I(\Lambda'). \tag{157}$$

REMARK 5.6. *If ρ, ρ' are W-topologically conjugate then they are topologically conjugate.*

REMARK 5.7. *If ρ, ρ' are Anosov then they are $\{e\}$-topologically conjugate.*

In particular, we may choose $F : \Lambda \to \Lambda'$ to be $\xi' \circ \xi^{-1}$, since the boundary map is continuous and dynamics-preserving. However, hyperconvex representations are $\{e\}$-topologically conjugate to non-hyperconvex Anosov representations, so $I = \{e\}$ isn't restrictive enough to have a relation to hyperconvexity.

We choose the ideal generated by all elements of length one. Define $I_\Delta = \{e, s_1, \ldots, s_n\}$.

THEOREM 5.8. *If $\rho, \rho' : \Gamma \to$ are I_Δ-topologically conjugate, then ρ is hyperconvex if and only if ρ' is hyperconvex.*

LEMMA 5.9. *Suppose F is an I_Δ-topological conjugacy between ρ and ρ'. Then there exists a permutation $\sigma \in Sym(n)$ such that for all distinct $x, y \in \Lambda$*

$$F(\alpha_k(x)) = \alpha_{\sigma(k)}(x'), \text{ and} \tag{158}$$

$$F(x_k^y) = x_{\sigma(k)}^{\prime y'}. \tag{159}$$

Recall that any hyperbolic group which admits hyperconvex representations is virtually Kleinian [**19**]. Two important classes of such Γ are those with boundary homeomorphic to $\mathbb{CP}^1$ or to S^1. We construct nontrivial I_Δ-topological conjugacies explicitly in both cases.

PROPOSITION 5.10. *Let $\rho, \rho' : \pi_1\Sigma \to SL_{n+1}\mathbb{C}$ be $SL_{n+1}\mathbb{C}$-quasi-Fuchsian. Then ρ and ρ' are I_Δ-topologically conjugate.*

PROPOSITION 5.11. *Let Γ be a word hyperbolic group such that $\partial\Gamma \cong \mathbb{CP}^1$. If $\rho, \rho' : \Gamma \to G$ are hyperconvex, then they are I_Δ-topologically conjugate.*

CONJECTURE 5.12. *If $\rho, \rho' : \Gamma \to G$ are hyperconvex, then they are I_Δ-topologically conjugate.*

If this conjecture is true, and if topological conjugacy classes are open among irreducible Anosov representations, then topological conjugacy classes could be used to differentiate between hyperconvex and non-hyperconvex Anosov components in $\mathrm{Hom}(\Gamma, G)$.

5. Proof of Theorem 5.8

We prove our statement about I_Δ-topological conjugacies assuming Lemma 5.9.

PROOF OF THEOREM 5.8. Let F be an I_Δ-topological conjugacy between ρ and ρ', and suppose ρ is hyperconvex.

Let $x, y, z \in \Lambda$ distinct, and let $1 \leq k \leq n$. By Lemma 5.9, there exists a permutation $\sigma \in \mathrm{Sym}(n)$ such that $F(x_k^y) = x'^{y'}_{\sigma(k)}$ and $F(x_k^z) = x'^{z'}_{\sigma(k)}$. In particular, since $x_k^y \neq x_k^z$ and F is a homeomorphism, then $x'^{y'}_{\sigma(k)} \neq x'^{z'}_{\sigma(k)}$.

Since k was arbitrary, then $x_k'^{y'} \neq x_k'^{z'}$ for all $1 \leq k \leq n$. By Prop. 4.2 then $\mathrm{Rich}^0_{x',y'} \cap \mathrm{Rich}^0_{x',z'} = \{x'\}$. Since $x, y, z \in \Lambda$ were arbitrary, then ρ' is hyperconvex.

The other direction proceeds similarly. □

6. Existence of partial topological conjugacies

Next, we prove the existence of I_Δ-topological conjugacies between hyperconvex representations of groups with boundaries $\mathbb{CP}^1$ or S^1.

PROOF OF PROPOSITION 5.11. Let Γ such that $\partial\Gamma \equiv \mathbb{CP}^1$. Let $\rho, \rho' : \Gamma \to G$ hyperconvex. Define F on $\Lambda \to \Lambda'$ to be $\xi' \circ \xi^{-1}$, which is a dynamics-preserving homeomorphism because ρ and ρ' are Anosov.

Let $p \in Th^{I_\Delta}(\Lambda) \setminus \Lambda$. Since by Corollary 4.10 $\pi_k^x : \Lambda \to \alpha_k(x)$ is surjective for all $x \in \Lambda$ and $1 \leq k \leq n$, then there exist unique $x, y \in \Lambda$ and $1 \leq k \leq n$ such that $p = x_k^y$. Let

$x' = F(x)$ and $y' = F(y)$, and define $F(p) = x_k'^{y'} \in \alpha_k(x') \subset Th^{I_\Delta}(\Lambda')$. Since p was arbitrary, we can define F on all of $Th^{I_\Delta}(\Lambda)$. Since ρ and ρ' are hyperconvex, then F is injective.

By Prop. 3.7 and Prop. 4.9, F is continuous. By Cor. 4.10, F is a surjection. It remains to show that F is equivariant. Let $\gamma \in \Gamma$. We have

$$F(\rho(\gamma) \cdot p) = F(\rho(\gamma) \cdot x_k^y) \tag{160}$$

$$= F((\rho(\gamma) \cdot x)_k^{\rho(\gamma)\cdot y}) \tag{161}$$

$$= (\rho'(\gamma) \cdot x')_k^{\rho'(\gamma)\cdot y'} \tag{162}$$

$$= \rho'(\gamma) \cdot (x_k'^{y'}) \tag{163}$$

$$= \rho'(\gamma) \cdot F(p). \tag{164}$$

So F is an I_Δ-topological conjugacy. □

Remark 5.13. *The topological conjugacy in the proof of Prop. 5.11 was not unique. In particular, let $\sigma \in Sym(n)$. We could have instead defined $F(p)$ to be $x_{\sigma(k)}'^{y'} \in \alpha_{\sigma(k)}(x)$ for all $p = x_k^y \in Th^{I_\Delta}(\Lambda)$.*

Proof of Proposition 5.10. Let $\rho, \rho' : \pi_1\Sigma \to SL_{n+1}\mathbb{C}$ be $SL_{n+1}\mathbb{C}$-quasi-Fuchsian. Then there exists an irreducible representation $i : SL_2\mathbb{C} \to SL_{n+1}\mathbb{C}$ and quasi-Fuchsian representations $\rho_0, \rho_0' : \pi_1\Sigma \to SL_2\mathbb{C}$ such that $\rho = i \circ \rho_0$ and $\rho' = i \circ \rho_0'$.

By Sullivan [**36**] then there is a topological conjugacy $F_0 : \mathbb{CP}^1 \to \mathbb{CP}^1$ between ρ_0 and ρ_0'. We use F_0 to construct F explicitly.

The Plucker embedding embeds $Gr(k, n+1)$ into $\mathbb{P}(\wedge^k\mathbb{C}^{n+1})$, the projectivized space of k-forms on $\mathbb{C}^{n+1}$ [**25**]. The space of k-forms of $\mathbb{C}^n$ is a complex projective vector space of dimension $\binom{n+1}{k}$. Similarly, $P(\mathbb{C}^D) \cong \mathrm{Sym}^{D-1}(\mathbb{CP}^1)$ where V^D is any D-dimensional complex vector space and $\mathrm{Sym}^{D-1}(\mathbb{CP}^1)$ is space of unordered $D-1$ tuples of elements of $\mathbb{CP}^1$. Define $d_k := \binom{n+1}{k} - 1$, and observe that $d_1 = d_n = n$ and $d_k = d_{n+1-k}$.

Thus we may identify $\mathcal{F}(\mathbb{C}^{n+1})$ with a subset of

$$\mathrm{Sym}^{d_1}(\mathbb{CP}^1) \times \cdots \times \mathrm{Sym}^{d_n}(\mathbb{CP}^1), \tag{165}$$

subject to additional incidence conditions. We use additive notation $x = x_1 + \dots x_{d_k}$ for elements of $\mathrm{Sym}^{d_k}(\mathbb{CP}^1)$. The irreducible embedding i induces an embedding $i : \mathbb{CP}^1 \to \mathcal{F}(\mathbb{C}^{n+1})$, and without loss of generality, we may assume that i sends $x_0 \in \mathbb{CP}^1$ to $(d_1 x_0, \dots, d_n x_0) \in \mathcal{F}(\mathbb{C}^{n+1})$.

Then for each $x_0 \in \mathbb{CP}^1$ and each $1 \leq k \leq n$ we have

$$\alpha_k(i(x_0)) = \{(\dots, d_{k-1}x_0, (d_k - 1)x_0 + p_0, d_{k+1}x_0, \dots) : p_0 \in \mathbb{CP}^1\}. \tag{166}$$

In particular, elements of $\alpha_k(i(x_0))$ differ from $i(x_0)$ only in the k-th index. Then

$$Th^{I_\Delta}(i(x_0)) = \cup_{k=1}^{n} \alpha_k(i(x_0)) \tag{167}$$

Define F on Λ to be $\xi' \circ \xi$, which is clearly continuous and dynamics-preserving. Then $\xi' \circ \xi^{-1} = i \circ \xi_0' \circ \xi_0^{-1} \circ i^{-1}$ since $\Lambda, \Lambda' \subset i(\mathbb{CP}^1) \subset \mathcal{F}(\mathbb{C}^{n+1})$.

Let $p \in Th^{I_\Delta}(\Lambda) \setminus \Lambda$. Then there exists $x_0 \in \Lambda_0$ and $1 \leq k \leq n$ such that $p \in \alpha_k(i(x_0))$. Then for some $p_0 \in \mathbb{CP}^1 \setminus x_0$ we have

$$p = (\dots, d_{k-1}x_0, (d_k - 1)x_0 + p_0, d_{k+1}x_0, \dots). \tag{168}$$

Using $x_0' = F_0(x_0)$ and $p_0' = F_0(p_0)$. Define $F(p)$ as follows:

$$F(p) = (\dots, d_{k-1}x_0', (d_k - 1)x_0' + p_0', d_{k+1}x_0', \dots). \tag{169}$$

Clearly F is injective, surjective, and continuous, because F_0 has these properties on $\mathbb{CP}^1$.

We confirm that F is a topological conjugacy by showing it is equivariant. Let $\gamma \in \Gamma$. Then

$$\rho'(\gamma) \cdot F(p) = \rho'(\gamma) \cdot (\dots, d_{k-1}x'_0, (d_k - 1)x'_0 + p'_0, d_{k+1}x'_0, \dots) \tag{170}$$

$$= \rho'(\gamma) \cdot (\dots, d_{k-1}F_0(x_0), (d_k - 1)F_0(x_0) + F_0(p_0), d_{k+1}F_0(x_0), \dots) \tag{171}$$

$$= F(\rho(\gamma) \cdot (\dots, d_{k-1}x_0, (d_k - 1)x_0 + p_0, d_{k+1}x_0, \dots)) \tag{172}$$

$$= F(\rho(\gamma) \cdot p). \tag{173}$$

Thus F is an I_Δ-topological conjugacy between ρ and ρ'. □

7. F preserves the Richardson 0-varieties of Λ

The remainder of this chapter is devoted to the proof of Lemmas 5.2 and 5.9. Assume $F : \mathcal{F} \to \mathcal{F}$ is a topological conjugacy between $\rho, \rho' : \Gamma \to G$. We use the notation $x' := F(x), y' := F(y), z' := F(z)$ for any $x, y, z \in \Lambda$.

First, we show that F preserves Richardson 0-varieties of pairs x, y which are the attracting and repelling flags of an infinite order element $\gamma \in \Gamma$.

PROPOSITION 5.14. *Let $\gamma \in \Gamma$ infinite order, and define $x = \xi(\gamma^+)$ and $y = \xi(\gamma^-)$. Then F preserves the Richardson 0-varieties of x and y.*

We observe that ρ is Anosov, so $\rho(\gamma)$ is proximal, and we can apply Theorem 3.8 and its corollary from Chapter 3.

PROOF OF PROPOSITION 5.14. By Cor. 3.9, the fixed points of $\rho(\gamma)$ acting on $\mathcal{F}$ are the Richardson 0-varieties of x and y, and the fixed points of $\rho'(\gamma)$ are the Richardson 0-varieties of x' and y'. Since F is a topological conjugacy, then it must send the fixed points of $\rho(\gamma)$ to those of $\rho'(\gamma)$, and therefore F preserves the Richardson 0-varieties of x and y. □

We then extend our argument in Proposition 5.14 from pairs $\xi(\gamma^+), \xi(\gamma^-)$ to all distinct pairs in Λ. The main ideas are that pairs of the form $\xi(\gamma^+), \xi(\gamma^-)$ are dense in Λ^2 (Proposition 2.6) and that $\rho(\Gamma)$ acts as a convergence group action on the limit set (Proposition 2.7).

We consider limit set pairs where only one flag is the attracting fixed point of an infinite order element.

PROPOSITION 5.15. *Let $x = \xi(\gamma^+)$ for some infinite order $\gamma \in \Gamma$, and let $z \neq x \in \Lambda$. Then F preserves the Richardson 0-varieties of x and z.*

PROOF OF PROPOSITION 5.15. Let x, z, and γ as given. Let $\{\gamma(i)\}_{i=1}^{\infty} \in \Gamma^{\mathbb{N}}$ as in Prop 2.7, so that $\lim_{i\to\infty} \rho(\gamma(i)) \cdot y = z$ for all $y \in \Lambda \setminus \{x\}$. In particular we may take $y = \xi(\gamma^{-1}) \in \Lambda \setminus \{x\}$. For all $i \in \mathbb{N}$, we use the following notation:

$$x(i) := \rho(\gamma(i)) \cdot x, \tag{174}$$

$$y(i) := \rho(\gamma(i)) \cdot y, \tag{175}$$

$$x'(i) := F(x(i)), \text{ and} \tag{176}$$

$$y'(i) := F(y(i)). \tag{177}$$

Observe that $\gamma(i)\gamma\gamma^{-1}(i) \in \Gamma$ is an infinite order element, and that $x(i), y(i)$ are the attracting and repelling fixed points of $\rho(\gamma(i)\gamma\gamma^{-1}(i))$. We may then apply Prop. 5.14 so that for all $i \in \mathbb{N}$, F preserves the Richardson 0-varieties of $x(i)$ and $y(i)$. By Prop. 3.7, then $\lim_{i\to\infty} \mathrm{Rich}^0_{x(i),y(i)} = \mathrm{Rich}^0_{x,z}$. By continuity of F, then $\lim_{i\to\infty} x'(i) = x'$ and $\lim_{i\to\infty} y'(i) = z'$. We apply Prop. 3.7 again to the sequence of transverse pairs $(x'(i), y'(i))$, so then

$$F(\mathrm{Rich}^0_{x,z}) = \lim_{i\to\infty} F(\mathrm{Rich}^0_{x(i),y(i)}) \tag{178}$$

$$= \lim_{i\to\infty} \mathrm{Rich}^0_{x'(i),y'(i)} \tag{179}$$

$$= \mathrm{Rich}^0_{x',z'}. \tag{180}$$

Thus F preserves the Richardson 0-varieties of x and z. □

Finally, we extend Proposition 5.15 to Richardson varieties of pairs y, z where both y and z are arbitrary distinct points of Λ.

PROPOSITION 5.16. *Let $y, z \in \Lambda$ distinct. Then F preserves the Richardson 0-varieties of y and z.*

PROOF. If y or z is the attracting fixed point of an infinite order element $\gamma \in \Gamma$, then we are done by Prop 5.15. Assume not.

Let $\{x(i)\}_{i=1}^{\infty} \in \Lambda^{\mathbb{N}}$ such that each $x(i) = \xi(\gamma(i)^+)$ is the attracting fixed point of an infinite order element, and $\lim_{i\to\infty} x(i) = y$. This sequence exists because attracting fixed points are dense in Λ by Prop. 2.6. Clearly $x(i) \neq z$, so they are transverse for all $i \in \mathbb{N}$. Define $x'(i) := F(x(i))$ for all $i \in \mathbb{N}$. By the continuity of F, then $\lim_{i\to\infty} x'(i) = y'$.

The argument proceeds similarly to that of Prop. 5.15. By Prop. 5.15, then for all $i \in \mathbb{N}$, F preserves the Richardson 0-varieties of $x(i)$ and z. By Prop. 3.7, then $\lim_{i\to\infty} \mathrm{Rich}^0_{x(i),z} = \mathrm{Rich}^0_{y,z}$. We apply Prop. 3.7 again to the transverse sequence $\lim_{i\to\infty}(x'(i), z') = (y', z')$, so then:

$$F(\mathrm{Rich}^0_{y,z}) = \lim_{i\to\infty} F(\mathrm{Rich}^0_{x(i),z}) \tag{181}$$

$$= \lim_{i\to\infty} \mathrm{Rich}^0_{x'(i),z'} \tag{182}$$

$$= \mathrm{Rich}^0_{y',z'}. \tag{183}$$

Thus F preserves the Richardson 0-varieties of Λ. □

8. F ϕ-matches the Richardson varieties of Λ

In this section, we show that F ϕ-matches all Richardson varieties of Λ, which completes the proof of Lemma 5.2.

PROPOSITION 5.17. *For $x, y \in \Lambda$ the attracting and repelling fixed points of some infinite order $\gamma \in \Gamma$, then there is a Bruhat order automorphism ϕ_{xy} such that F ϕ_{xy}-matches the Richardson varieties of x and y.*

PROOF. Since F preserves the Richardson 0-varieties of x and y, then there is a set automorphism $\phi_{xy} : W \to W$ such that $F(x^y_w) = x'^{y'}_{\phi_{xy}(w)}$.

We claim that ϕ_{xy} is a Bruhat order automorphism. Let $u \geq v \in W$. Then $\alpha_u(x) \cap \alpha_{\iota(v)}(y)$ is a nonempty irreducible complex projective variety of dimension $l(v) - l(u)$. In particular, by Thm. 3.8, a constructable open subset $U_{uv} \subset \alpha_u(x) \cap \alpha_{\iota(v)}(y)$ (consisting of elements not contained in a smaller Richardson subvariety) has attracting fixed point x_v^y and repelling fixed point x_u^y. Then since F is a topological conjugacy, $F(U_{uv})$ is the dense, open subset with attracting fixed point $x'^{y'}_{\phi_{xy}(v)} = F(x_v^y)$ and repelling fixed point $x'^{y'}_{\phi_{xy}(u)} = F(x_u^y)$. So $F(U_{uv}) \subset \alpha_{\phi_{xy}(u)}(x') \cap \alpha_{\iota(\phi_{xy}(v))}(y')$, meaning that the Richardson variety is nonempty, and in particular, $\phi_{xy}(u) \geq \phi_{xy}(v)$. So ϕ_γ is a Bruhat order automorphism.

Furthermore, observe that the Zariski closure of U_{uv} is $\overline{U_{uv}} = \alpha_v(x) \cap \alpha_u(y)$, and F is closed, so

$$F(\alpha_v(x) \cap \alpha_{\iota(u)}(y)) = F(\overline{U_{uv}}) \tag{184}$$

$$= \overline{F(U_{uv})} \tag{185}$$

$$= \alpha_{\phi_{xy}(v)}(x') \cap \alpha_{\phi xy(u)}(y'). \tag{186}$$

In particular, since all Richardson varieties of x and y are of the form $\alpha_v(x) \cap \alpha_{\iota(u)}(y)$ for $u \leq v \in W$, then F preserves and ϕ_{xy}-matches the Richardson varieties of x and y. □

PROPOSITION 5.18. *There exists a Bruhat order automorphism ϕ so that for all distinct, rational $x, z \in \Lambda$, F ϕ-matches the Richardson varieties of x and y.*

PROOF. Let $x, z \in \Lambda$ be distinct and rational. Then

Let $\gamma, \mu \in \Gamma$ infinite order elements. Define $x = \xi(\gamma^+)$, $z = \xi(\mu^+)$, $x' = F(x)$, and $z' = F(z)$.

Let $w \in W$. Consider $x_w^z \in \text{Rich}_{x,z}^0$. Then $F(\alpha_w(x)) = \alpha_{\phi_x}(x)$, because F ϕ_x-matches the Richardson varieties of x and $\xi(\gamma^-)$, and similarly $F(\alpha_{\iota(w)}(z)) = \alpha_{\phi_z(\iota(w))}(z') = \alpha_{\iota(\phi_z(w))}(z')$. Then

$$F(x_w^z) \in F(\alpha_w(x) \cap \alpha_{\iota(w)}(z)) \tag{187}$$

$$= \alpha_{\phi_x(w)}(x') \cap \alpha_{\iota(\phi_z(w))}(z'). \tag{188}$$

In particular, $\phi_x(w) \geq \phi_z(w)$. Both ϕ_x and ϕ_z are Bruhat order automorphisms, so $l(\phi_x(w)) = l(\phi_z(w)) = l(w)$, which means $\phi_x(w) \geq \phi_z(w)$ implies that $\phi_x(w) = \phi_z(w)$ Since w was arbitrary, we have $\phi_x = \phi_z$. □

PROPOSITION 5.19. *There is a global Bruhat order automorphism ϕ so that for all distinct $x, y \in \Lambda$, F ϕ-matches the Richardson varieties of x and y.*

PROOF. Let ϕ be the automorphism from Proposition 5.18. Define $V \subset \Lambda$ to be the attracting fixed points of infinite order $\gamma \in \Gamma$, and $V' = F(V)$. Observe that by the continuity of the Richardson 0-varieties, and the density of V in Λ (Prop. 2.6), then F ϕ-matches the 0-Richardson varieties of all distinct pairs in Λ.

Let $u \in W$. Then $\cup_{x\in\Lambda}\alpha_u(x)$ is a closed subset, and $F(\cup_{x\in V}\alpha_u(x)) = \cup_{x'\in V'}\alpha_{\phi(u)}(x')$. In particular, by the continuity of F, then

$$F(\cup_{x\in\Lambda}\alpha_u(x)) = \overline{F(\cup_{x\in V}\alpha_u(x))} \tag{189}$$

$$= \overline{\cup_{x'\in V'}\alpha_{\phi(u)}(x')} \tag{190}$$

$$= \cup_{x'\in\Lambda'}\alpha_{\phi(u)}(x'). \tag{191}$$

Because $\mathcal{F}$ is Hausdorff, and F is continuous, then $F(\alpha_u(x)) = \alpha_{\phi(u)}(x')$ for all $x \in \Lambda$.

Then F necessarily ϕ-matches the Richardson varieties of Λ, since $F(\alpha_w(x) \cap \alpha_u(y)) = \alpha_{\phi(w)}(x') \cap \alpha_{\phi(u)}(y')$ for all $w, u \in W$ and all $x, y \in \Lambda$ distinct. □

This concludes the proof of Lemma 5.2.

PROOF OF LEMMA 5.9. The argument mostly follows that of Lemma 5.2. Define $I := I_\Delta$.

Let $\gamma \in \Gamma$ be an infinite order element. Then $\rho(\gamma)$ is proximal. Then for $x = \xi(\gamma^+)$ and $y = \xi(\gamma^-)$ then by Prop. 3.16 x_k^y is a fixed point of $\rho(\gamma)$ for all $1 \leq k \leq n$.

Then we have

$$F(\{x_1^y, \ldots, x_n^y\}) = \{x_1'^{y'}, \ldots, x_n'^{y'}\}. \tag{192}$$

Such pairs x, y are dense in Λ^2 and by the convergence action of Γ (Prop. 2.7) and the continuity of Richardson varieties (Prop. 3.7) then for all distinct pairs $x, y \in \Lambda$ we have

$$F(\{x_1^y, \ldots, x_n^y\}) = \{x_1'^{y'}, \ldots, x_n'^{y'}\}. \tag{193}$$

So for each pair x, y there is a permutation $\sigma_{xy} \in \mathrm{Sym}(n)$ such that $F(x_k^y) = x'^{y'}_{\sigma_{xy}(k)}$. By the continuity of F, $F(\alpha_k(x)) = \alpha_{\sigma_{xy}(k)}(x')$, so the permutation σ_{xy} does not depend on y. Similarly, by the continuity of F, $F(\pi_k^\Lambda(y)) = \pi^{\Lambda'}_{\sigma_{xy}(k)}(y')$ so the permutation σ_{xy} does not depend on x. Thus all permutations σ_{xy} must be equal, so $\sigma = \sigma_{xy}$ is global. $\square$

Statement on the usage of generative AI

No generative AI was used in the research or writing of this thesis.

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Curriculum Vitae

8.1. Education.

- 2022 University of Illinois at Chicago, MS in Mathematics
- 2017 Olin College of Engineering, BS in Electrical and Computer Engineering

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- 2020-2026 University of Illinois Chicago, Department of Mathematics, Statistics, and Computer Science
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- Summer 2022 Los Alamos National Laboratory, Computational Physics Graduate Researcher
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8.3. Publications.

- **Vaccaro, Jennifer** and Konstantin Lipnikov. "Applying an Oriented Divergence Theorem to Swept Face Remap." SIAM J. Numer. Anal. 61 (2023): 2285-2304.
- Vaughn, Ian, Stefano Suman, Zachary Berkowitz, **Jennifer Vaccaro**, et. al. "Transitioning to Open Source at 6000m." 2018 IEEE/OES Autonomous Underwater Vehicle Workshop (AUV) (2018): 1-6.
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- 2025 "Math Meets Policy: Advocating for Change." Joint Mathematics Meetings AMS-MAA, Seattle, WA.
- 2024 "Higher rank symmetric spaces." 20 years of Anosov representations, MPI Leipzig, Germany.

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- Fall 2026 Math 088 Intermediate Algebra Workshop (Instructor)
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